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ABSTRACT

The main purposes of the seminar were to examine the present status of school mathematics problem solving; to explore classroom practice and what is known about research; to compare the situations appearing in textbook word problems; and to make plans for cross-cultural research in both countries. A paper from each country was presented during the eight sessions. The session topics were: (1) "The Present States of Problem Solving in the U.S. and Japan"; (2) "Classroom Practice of Problem Solving"; (3) "Comparison of Achievement"; (4) "Comparison of Situations Appearing in Word Problems"; (5) "Typical Word Problem Solving"; (6) "Pattern Finding"; (7) "Non-Routine Problem Solving"; and (8) "Mathematical Modelling." The closing session was an open discussion on the formulation of the seminar and joint cross-cultural research and the formation of a joint study group. A discussion of the findings and proposals is given, and an appendix includes examples of problems in mathematics for the entrance examinations to national and private universities in Japan. (YP)

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Proceedings of the U.S.-Japan Seminar on Mathematical Problem Solving

Editors

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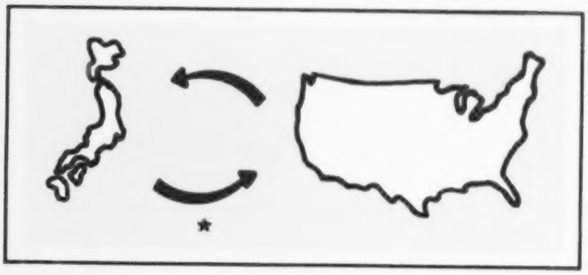

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PROCEEDINGS OF THE U.S.-JAPAN SEMINAR OF MATHEMATICAL PROBLEM SOLVING

Edited by

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July 14-18, 1986

at

The East-West Center, Honolulu, Hawaii

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In the Japanese Garden of the East-West Center

Back row: S. Yoshikawa, E. Silver, T. Sawada, S. Clarkson, S. Rachlin, N. Whitman, J. Wilson, K. Travers
Middle row: K. Inouye, N. Nohda, H. Senuma, M. G. Kantowski, B. Travis, T. Ishida, Y. Hashimoto
Front row: E. Nagasaki, Y. Sugiyama, S. Shimada, T. Miwa, J. Becker
(Kneeling)

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PREFACE

Jerry Becker and Tatsuro Miwa, Co-Organizers

These are the Proceedings of the U.S.-Japan Seminar on Mathematical Problem Solving held at the East-West Center in Honolulu, Hawaii July 14-18, 1986. The Seminar and these Proceedings mark the importance placed on problem solving in school mathematical education in both the United States and Japan during the decade of the 1980's.

We believe, along with all the delegates, that the Seminar was a success. Interesting, scholarly papers and discussions filled the Seminar Agenda. It was also an enjoyable event held in the superb facilities of the East-West Center with the beautiful Japanese Garden in the background. Not only did the participants find the Seminar valuable, but the event marked the mutual and increasing interest by mathematics educators from both countries in extending communication, exchange and cross-cultural collaboration in research.

We want to extend our heartiest appreciation to all the delegates who, through their paper presentations and discussion, accounted for so much of the quality interaction during the Seminar. We need to also express appreciation to the National Science Foundation (NSF) and the Japan Society For the Promotion of Science (JSPS) which, through the U.S.-Japan Cooperative Science Program, made this Seminar possible. In particular, Dr. Charles Wallace (NSF) has our thanks for his important role in the Seminar's success.

No bi-national seminar can be successful without competent translators. In this respect, the Seminar was exceedingly fortunate to have Dr. James Kenney and Mr. Kenji Inouye as translators. Not only were they highly knowledgeable about the intracacies of translating between English and Japanese, but they were friendly, amiable individuals who cooperatively worked patiently and tirelessly to smooth communication during Seminar sessions and social activities. To both we extend our profound appreciation.

As mentioned above, the facilities of the East-West Center are superb and certainly they were ideal for our Seminar. For providing comfortable meeting arrangements, an excellent technical setup, and a staff of friendly and supportive individuals, we need to convey our deep appreciation to Mr. James McMahon, the Logistics Officer of the East-West Center. Through Mr. McMahon's support and patience, our Seminar was helped to success. To members of his staff goes our sincere thanks: Ms. Rowena Kumabe, Ms. Norma Heen, Mr. Marshall Kingsbury, Ms. Margo Shiroyama, Ms. Leigh Hamasaki, Ms. Tammy Lewis, Ms. Noela Napoleon, and Ms. Cassandra Olayvar.

This Seminar was an important one and perhaps it is useful to describe its origin. The Co-Organizers first met at the U.S.-Japan Seminar on Mathematics Education held at the National Institute For Educational Research in Tokyo in April, 1971. In the interim we had communication and met again at the ICME-JSME Regional Conference held in Tokyo in October, 1983. At that time, we engaged in conversations with Professor Shigeru Shimada and others about the appropriateness, timing, and content of another meeting that would deal with problem solving. Later, further discussion was held with Professor James Wilson. All agreed that such a Seminar would be useful, as well as timely, and it was decided to seek support by submitting proposals simultaneously to the NSF and JSPS. These proposals were reviewed on both sides and recommended for support. There ensued preparation on both sides covering a time period of 1-2 years, culminating in our Seminar at the East-West Center.

Finally, we express our appreciation to Dr. Art King, Director of the Curriculum Research and Development Group at the University of Hawaii, the Dean of the College of Education, Dr. John Dolly, and Dr. Loretta Krause, Principal of the University Lab School. They received us and made our visits outside the Seminar interesting and rewarding. To Ms. Carole Shirley goes our thanks for transcribing all Seminar discussions, and to Ms. Joan Griffin our heartfelt appreciation for her enormous energy and friendly competence in typing these Proceedings into final form. Ms. Griffin and Ms. Lynda Hurley, along with Ms. Pat Brey, assisted in proofreading these proceedings, though any mistakes are the responsibility of the Editors.

It is our earnest hope that these Proceedings will be of interest to mathematics educators in both countries, as well as to others who share our desire to advance the cause of an improved mathematics education for children and students at all school levels.

Jerry P. Becker Tatsuro Miwa

August, 1987

Special Note: The American delegation was pleased to host the Seminar in Honolulu. We wish to include a special acknowledgement to members of the Japanese delegation. All Japanese delegates prepared and presented their papers in English in excellent fashion. This represented a significant effort on their part, an effort for which we are deeply appreciative.

Jerry P. Becker

SEMINAR PURPOSES AND PROCEDURES

There has been considerable interest and a large number of activities in mathematics education in both Japan and the United States in recent years. Mathematics educators in both countries are exploring ways in which student achievement can be improved in all areas of school mathematics. But the area of greatest concern and the area in which mathematics educators of both countries are focusing their attention is problem solving. Accordingly, this is the area of focus for the present Seminar and subsequent research.

During the discussions between American and Japanese mathematics educators, starting in 1983 and continuing through 1985, a great and mutual interest was expressed in bringing mathematics educators on both sides together to improve communication and propose further research. A Joint U.S.-Japan Seminar seemed like an excellent manner by which to do this. The main purposes of the Seminar were set as follows:

1. to examine the present state of problem solving in school mathematics in the United States and Japan
2. to explore classroom practices in problem solving in the United State and Japan
3. to examine existing data concerning problem solving in the two countries
4. to compare the situations appearing in typical textbook word problems in both countries
5. to explore what is known about research in each country relating to the pattern-finding behavior of students in problem solving
6. to explore what is known about research in each country relating to the mathematical model-making behavior of students in problem solving.
7. to make plans for cross-cultural research in problem solving in both countries, including:
    - * identifying specific problems that would be used in the research
    - * identifying the levels at which data will be gathered
    - * identifying the instruments/research approaches to be used in the studies
    - * gathering common data for the two countries
    - * exchanging and analyzing data
    - * reporting research results to the larger mathematics education communities in both countries, as well as the larger international community

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OPENING SESSION

Becker: The Joint Japan-United States Seminar On Cross-Cultural Research On Students' Strategies Dealing With Mathematical Problem Solving is about to begin. Members of both the American and Japanese delegations are now gathered here in Hawaii in the middle of the Pacific Ocean for the Seminar. On behalf of the American delegation, I would like to offer our most sincere greetings to Professor Miwa and all members of the Japanese delegation.

Miwa: On behalf of the Japanese delegation, I would like to offer our most sincere greetings to members of the American delegation.

The idea for the Seminar began with several members of the Japanese delegation, including me, and Professor Becker, the Co-Organizer of the Seminar, who had come to Japan for the ICMI-JSME Conference on Mathematical Education in October, 1983. Following that first meeting, we had many more meetings and communications. The application (proposal) for the Seminar was submitted to JSPS and NSF in May of 1985, and approval was given in the end of last November. In the seven subsequent months, the American side officially drafted Professor Becker, whose devoted attention to the various fine points of the Seminar has made this meeting possible. All Japanese members and I are very grateful for the remarkable efforts made to organize the Seminar.

The importance of mathematics education in a democratic society is widely recognized today, and there is no need for me to elaborate on this point. And problem solving is now an issue of utmost immediacy for both Japan and the United States.

This Seminar will deal with problem solving from a cross-cultural standpoint, which is a very new approach to research, and I expect that there will be prime results arising from our deliberations. In order that there be a large success, I ask for the cooperation of all of you.

At this point, I would like to offer my thanks to the East-West Center which has provided the seminar room, and especially to Mr. James McMahon who has done so much to make the conference possible. In thinking about the Seminar with its focus on comparative cultural matters, the East-West Center is, perhaps, the most appropriate place for our meeting. Finally, I would like to express my thanks to both our translators, Dr. Kenney and Mr. Inouye. Our hope is that they will assist in lowering the very high language barrier and contribute towards the success of the Seminar. Thank you very much.

Becker: Phase one of the Seminar is now complete. The papers are written and we are all here. We now begin phase two which is perhaps the most important part of the whole process. Now we study the papers, discuss them, and do our best to communicate with each other about the content of the papers. Among our objectives are: (1) for each delegation to learn more about mathematics education in our two countries; (2) to learn more about research in mathematics education in our two countries; (3) to see what we can take from the papers and discussions that will help us understand the problem solving behavior of students and help to improve classroom teaching; and (4) to plan for future cooperative research that builds on the basis laid here at this Seminar.

Professor Miwa and I wish to thank all the participants for their diligent work in preparing their papers. I want to thank Professor Miwa for his consistently excellent cooperation as leader of the Japanese delegation in finalizing all the details, both conceptual and logistical, that have now brought

us together here at this beautiful meeting place. On behalf of the American delegation, I wish to welcome all of our Japanese colleagues to the U.S. and to the Seminar. It is our honor to have you with us. Now let us begin the work of the Seminar, which I feel will be historic in building interaction and cooperative efforts among Japanese and U.S. mathematics educators. Thank you.

SESSION I

Professor Shimada's Paper

Becker: It is my honor to begin the proceedings by introducing Professor Shigeru Shimada. I will make just a few remarks. I have known Professor Shimada for many years as a colleague in mathematics education and as a friend. He has held very important positions in Japanese mathematics education over a period of many years. Since 1982 he has been Professor of Mathematics Education on the Faculty of Science at the Science University of Tokyo. He has had experience as a mathematics teacher in the middle school attached to Tokyo Higher Normal School. He has also been a Specialist Officer in the Ministry of Education as well as Senior Researcher in the National Institute For Educational Research. Professor Shimada was also on the Faculty of Education at Yokohama National University. So, Professor Shimada, we welcome you and look forward to your talk.

Shimada: Thank you Mr. Chairman and ladies and gentlemen. I think it is my great honor to be assigned as the first speaker in this interesting Seminar. I would like to take this opportunity to express my sincere gratitude to Professors Miwa and Becker for their great leadership and efforts to organize this meeting.

Now I will go into my paper.

PROBLEM-SOLVING - THE PRESENT STATE AND HISTORICAL DEVELOPMENT IN JAPAN

Shigeru Shimada
Faculty of Science
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1. Introduction

Many papers on problem-solving in mathematics education seem to suggest that most difficulties pupils encounter in problem-solving are common to both countries, though the relative degree of difficulty may differ in some respects. But sometimes a delicate difference may be noticed. For example, the reversal mistake discussed by Resnick and Clement is not so common in Japan, at least in the upper secondary grades or at the college level. This kind of reflection motivated me to the consideration of factors which may affect positively or negatively on pupils' problem-solving behavior as well as teacher's behavior of helping them in the mathematics classroom and at the same time seem to be so ordinary in that society that observers and those observed both likely fail to become aware of them. Being ordinary in that society means that they constitute part of the socio-cultural tradition of general citizens and of teachers specifically.

I would like to enumerate these factors with a brief sketch of the historical development of mathematics teaching in Japan, in which these traditions were gradually formed. Because time is limited, most examples are omitted from my text, but, instead, are given in the Appendices, of which B to D were prepared by my collaborator, Mr. Eizo Nagasaki.

2. Adoption of Western Mathematics

Mathematics presently taught in the schools of many countries is international in its paradigm, structure, and symbolic system, though it is described by a variety of languages. But historically speaking, this is a product of Western civilization, originating in the Oriental and Greek civilizations and being formed through the modern development of Europe.

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Western mathematics was fragmentally imported into Japan beginning in the 18th century. After opening the country to foreign intercourse in 1858, a few pioneering scholars, naval and other military officers began to study it eagerly and systematically for the purpose of national defense. The new government took over the old regime of Tokugawa in 1868 (an event which is called the Meiji Restoration) and began to establish a modern school system throughout the country in 1872. Western mathematics was adopted as a school subject from the first year of elementary school. This adoption was a very bold decision because in those days there were only a handful of persons all over the country who were competent enough to teach the Western mathematics while there being much more persons who were able to teach a traditional Japanese mathematics, if allowed. This decision had a profound effect on the further development of Japan.

Traditional Japanese mathematics originated in China and was imported from there to Japan in 7th and 17th century. It developed independently to a greater extent during 17-19th centuries. Books for its elementary part, whose intended audience were adults in business or civil engineering, were first published in Japanese in the 17th century and gradually became popular through many revised or imitated versions. These contributed to diffusion of a basic knowledge of mathematics. Their contents were composed of examples showing the use of the abacus in multiplication and division and of real and practical problem-solving with notes for answers and method. They were supplemented with a few fictitious problems to meet peoples' intellectual curiosity in the form of games, quizzes and/or diversions.

Books for an advanced part were composed of problem-solving mainly concerned with computation related to geometrical figures and did not have any link with technology or the sciences. Development of themes was based on intuition and induction but not on deductive arguments. Competition for solving more difficult problems motivated mathematicians. One merit of this mathematics was that it provided a background in which Western mathematics was implanted together with some technical terms.

The laymen's view of mathematics was formed by this elementary version of Japanese mathematics during the 17-19th centuries. It seemed to remain as the attitude towards learning mathematics through several generations in the context of school education. Mathematics, especially its elementary part (arithmetic), was regarded as a tool for solving real-world problems and as an intellectual game with mysterious tricks, but not as an organized body of knowledge. This attitude tended to encourage a fragmented learning of skills in problem-solving by separating them into types.

Many efforts were needed to bring about the bold decision to adopt Western mathematics. But through about twenty years of struggle, a proto-type Japanese version of Western mathematics was established by the end of 19th century. Among others, the following adjustment measures to accomodate Western mathematics to the Japanese context seem relevant to our interests here.

While new Japanese terms corresponding to Western technical terms (having no equivalence in existing Japanese vocabulary) were devised or created, Western symbolic expressions, including the use of Arabic numerals and alphabets, were adopted with only small modifications. For example, the order for writing the multiplier and multiplicand in multiplication was reversed.

How to read aloud those symbolic expressions was a perplexing problem for both teachers and pupils. An informal solution was adopted in which symbols were read in Japanese (in most cases) one-by-one in the order of their appearance. Though this approach was not in accordance with Japanese syntax, through frequent repetition in school it gradually became recognized as a Japanese dialect in talking about mathematics.

Japanese sentences are traditionally written in vertical form from top to bottom and from right to left. This system is still widely used in daily life and also in non-scientific publications. But this is very inconvenient in mathematics when the Western symbolic system is used. Accordingly, our ancestors created a new form for writing Japanese in which sentences are written horizontally from left to right and lines from top to bottom, consistent with Western languages (See Appendix B).

It should be noted that symbolic expression may have the effect of shorthand writing of ordinary sentences in Western languages, but such is not the case in Japanese. Furthermore, in the Western context alphabets provide for abbreviation of nouns, but in Japanese this is not the case.

It is an arbitrary symbol with no specific meaning by itself, though repeated use of x, y, etc., is likely to cause a misunderstanding that they are symbols for unknown quantities by themselves. (Most Chinese characters have their own meaning, whereas 'kana' letters are only phonogram. In some cases, pupils may interpret alphabet as a kind of ideogram meaning unknown or known.)

Use of the horizontal writing system may give pupils a kind of readiness for mathematical context, and use of the Western system of symbols within the Japanese context may give an effect of 'background and figure' as in the theory of Gestalt psychology. At the same time, this may cause some different difficulties not experienced by Western children.

3. Pressure of Entrance Examination

A strong trend after the Meiji Restoration was a desire of people to climb the social ladder through schooling. This became possible for ordinary citizens regardless of their parents' status, provided they could afford it. Thus, entering an institution of higher education came to be regarded as a necessary means to realize this end. Because there were never enough positions in higher education for all who wanted to enter, there ensued a severe competition in the form of entrance examinations. Mathematics played a key role in this selection process because of the nature of its clear distinction between right and wrong and its ease in preparing problems of various levels of difficulty. This situation continued from the end of the 19th century right up to the present. Before the 2nd World War it was from elementary to secondary and secondary to tertiary, and following the War from lower secondary to upper secondary and upper secondary to tertiary.

Preparing pupils for examinations was considered a job for classroom teachers, and thus exerted a strong influence on their teaching at all levels. As preparation of pupils for examinations took on greater importance, the usual textbook problems provided little discriminating power and artificial and well-but complexly-structured problems were devised by examiners. These, in turn, were followed by teachers in their classes. Then examiners would seek to devise newer ones and a kind of see-saw game was played between them. As a way for preparing pupils for examination, teachers devised a classification of problems by type so that pupils could memorize the types to use in solving problems (see Appendix C for examples of types). This trend was especially prominent in the case of elementary schools before the 1930's.

As for the pupils, this tended to spur a narrow-minded preparation and fostered an attitude of hurrying to get an answer without reflection on the solving process. However, at the same time it provided good practice.

Examiners were supposed to expect examinees to write their solutions in a neat and systematic manner for ease of scoring. In order to meet this requirement, teachers encouraged their pupils to use a standard tyle of answering word-problems in the elementary school level. The style consisted of (1) expression in horizontal form, (2) a few words of note, (3) computation, and (4) a complete answer (e.g., omission of the unit would be given a reduced score). Writing expressions before computation thus became the usual classroom practice even when the teaching did not emphasize the preparation. This custom seems to help pupils to transfer from arithmetic to algebra. In cases where the use of equations is helpful in problem-solving, formulating the equation from verbally stated conditions usually requires expressing some quantity with two or more operations on given data and unknowns.

As time passed, it became clear that the approach of classifying problems was not so successful as expected; accordingly, two trends emerged: one was to teach a general strategy to attack a new problem (emphasized mainly in the upper grades), and the other was to teach the meaning of operations in a more effective manner (emphasized mainly in the lower grades). For the former, several strategies or tactics were advocated and published as books for use by pupils in preparing for examinations (see Appendix D for examples of such strategies). These proved helpful for some pupils. For the latter, teachers formulated a set of problems for use in introducing new concepts while making sure of their effectiveness by analysing their mathematical meaning and developmental stages (much like Freudenthal's phenomenology).

4. Further Development

While the school system, together with its syllabi and style of presentation (including those of mathematics), was fully established and expanded during 1900-1930, new movements to improve instruction in the schools gradually emerged from various sectors of education. The practice of praparation for examinations also became a matter of social concern. The pragmatic and child-centered philosophy of education entered into the elementary schools, and the improvement of mathematics teaching at the secondary school level (advocated by Perry, Klein, and Moore) began to have an influence on mathematics teachers in our country. Examination-based problems and problems for adult's real life (rudiments of so-called commercial mathematics) were criticized because they were unsuitable for teaching with respect to their situation, structure, naturalness, or motivational value. Progressive educators advocated the use of more pragmatic and real-to-child problems as well as problems in a broader sense such as half-or unstructured ones.

Along with these trends, which might be regarded as a reflection of Western thought, an element of traditional cultural thought seemed to re-emerge in a somewhat subtle way behind the movement. It is a tradition to seek a meta-physical or meta-technical mental attitude behind every kind of arts, techniques, or disciplines. Thus, as a basis for teaching mathematics, such words as scientific spirit, idea of function, and mathematical ways of thinking came to be used frequently in discussions. Further, the name for arithmetic was changed from 'sanjutsu' (meaning the 'art of counting') to 'sansu' (meaning arithmetic and mathematics) in 1941 in order to avoid a nuance indicated by use of 'jutsu' which means a technique or an art.

Concurrence of these two trends resulted in a new curriculum in the form of a series of textbooks for the elementary schools in 1935, and later secondary schools in 1941. The basic underlying philosophy of this new curriculum may be interpreted as identifying the learning process with the process of problem solving in its broader sense. A course of learning was considered as a process of problem solving as reflected in the following:

a new problem → development of conceptual tool to solve it →
solution → refinement and generalization of tool → a next problem

In this approach, new types of problems from the real or physical world were devised and mathematical diversions from the old Japanese mathematics were revived and incorporated in order to stimulate pupils' curiosity.

This approach continued to the end of the 1950's, though curricula were revised during this period, its intention could not be realized in classroom practice for many reasons (one of the main criticisms was its failure in the systematic development of topics). After the 1950's, the curriculum returned to that organized by a systematic sequence of subject-matters, though problem-solving activities in a rather restricted sense were also emphasized as one of objectives. In responding to public criticism against a decreasing level of pupil achievement, several strategies for problem-solving such as drawing a 'structure diagram' were developed.

The so-called "New Mathematics" was introduced in schools in the 1970's and emphasis shifted to a more conceptual kind of learning than a problem-solving one, though the value of good problems for development of mathematics was emphasized by eminent mathematicians. After introduction of the "New Mathematics," public criticism against 'too early introduction of abstraction and new terms and symbols' became severe. Consequently, in a further revision of curricula a formal introduction of new mathematics was retreated and the emphasis was gradually shifted toward teaching mathematical ways of thinking which form the basis of those modern concepts of mathematics. This may be regarded as a new form or appearance of the traditional cultural thought mentioned earlier.

5. Concluding Remarks

The following four socio-cultural factors which may have influence on pupils' behavior in problem solving are discussed in this paper:

1. The layman's view of mathematics as a tool for solving problems.
2. Use of the Western system of mathematics symbols incorporated in a horizontal way of writing Japanese sentences.

3. The pressure of entrance examinations, and its effects on school practice.
4. Teachers' emphasis on mathematical ways of thinking.

It would be difficult to clarify empirically the extent to which these factors may contribute to the outcome of pupils' learning on problem-solving in mathematics. However, if we want to make meaningful comparisons of different socio-cultural backgrounds, these should be taken into consideration in the design and analysis of research studies.

Appendix A

Aspects of a Problem

Since the word 'problem' is used in a variety of ways according to context and is sometimes likely to cause confusion in discussion, it would be useful to identify some aspects of 'problem' which may have different implications for mathematics teaching. The following aspects seem to be important in considering the educational value of problems, though reference is not made to all of them in the text of this presentation.

A. Proposer, i.e., who proposed it?

(1) pupils. (2) teachers. (3) outside adults such as author of textbooks or examiner.

B. Situation, i.e., with what kind of world is it concerned?

(1) pupil's daily life. (2) adult's social or economic life.
(3) physical world. (4) fictitious world or imaginary real world.
(5) conceptual world (of pure mathematics or other sciences.)

C. Structure, i.e., how is it structured?

(1) well and simply structured. (2) well but complexly structured.
(3) half- structured in terms of data or condition.
(4) unstructured, only goal and obstacle are specified.

D. Naturalness, i.e., is it a natural question in that situation?

(1) natural, likely to happen. (2) artificial, unlikely to happen.

E. Motivation, i.e., why must it be solved?

(1) practical. (2) emotional (to keep pace or compete with peers, or to please teachers or parents). (3) scientific (to satisfy intellectual curiosity or academic interest, or to meet needs in other fields of interest). (4) instrumental (it must be done to get a certificate, to pass an examination, or to get approval to do other things.)

From the teacher's point of view, problems may be classified by purposes of using them in teaching. That is:

P1. Introducing a new topic.

P2. Exercising learned concepts.

P3. Enriching meaning of learned concepts.

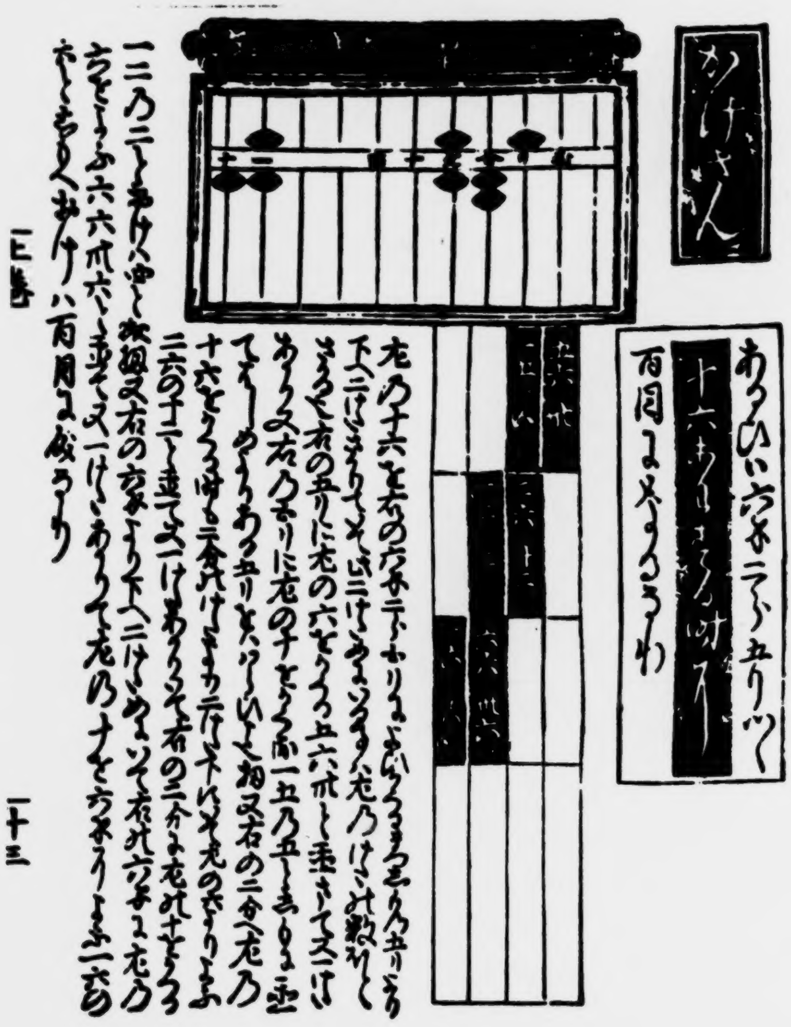
P4. Teaching strategy or tactics for problem solving.

APPENDIX B

Sample Pages From Old Japanese Mathematics Books To Show Different Writing Systems.

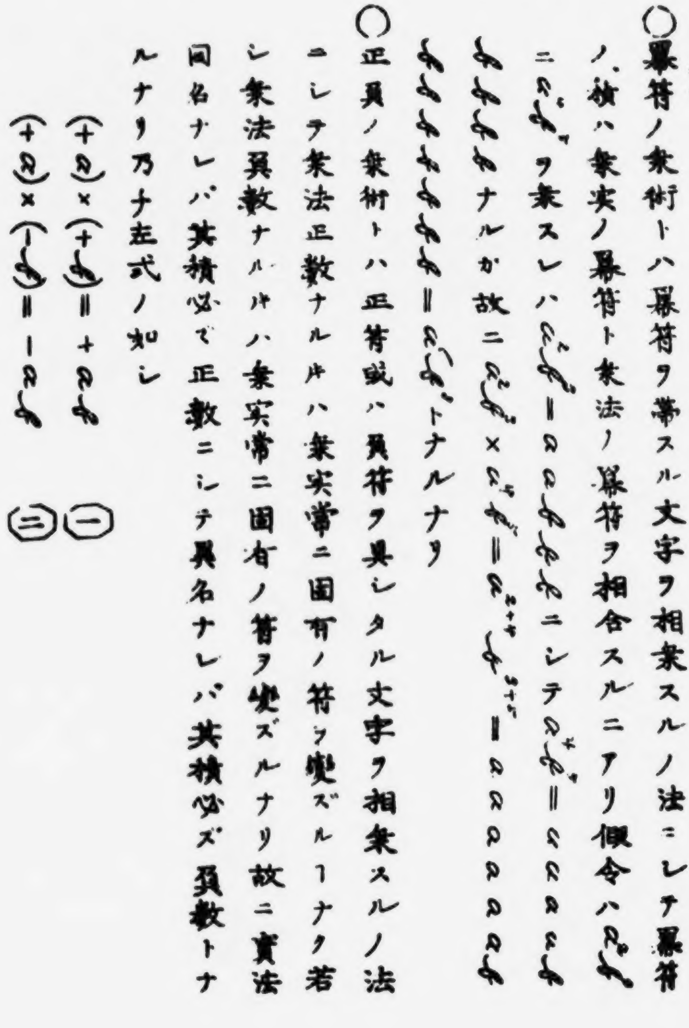
(1) Figure 1 shows a page (17cm x 27cm) from one of the most popular elementary mathematics books, Jinkoki, by M. Yoshida, published in 1627. Sentences are written vertically with an illustration of abacus using for 16 x 6.25 = 160.00 and a table to show what multiplication facts are used.

Figure 1



(2) Figure 2 shows a page (12cm x 22cm) from one of early publications on Western algebra, Daisushinsho by Y. Nishida, published in 1877. Japanese sentences are written vertically and Western symbolic expressions are inserted among them with a quarter turn.

Figure 2



○冪符ノ乗術トハ冪符ヲ帯スル文字ヲ相乗スルノ法ニシテ冪符ノ積ハ乗実ノ冪符ト乗法ノ冪符ヲ相合スルニアリ假令ハ$a^{二}b^{三}$ニ$a^{三}b^{四}$ヲ乗スレハ$a^{二}b^{三}=aabbb$ニシテ$a^{三}b^{四}=aaabbbb$ナルガ故ニ$a^{二}b^{三}\times a^{三}b^{四}=a^{二+三}b^{三+四}=aaaaabbbbbbb=a^{五}b^{七}$トナルナリ

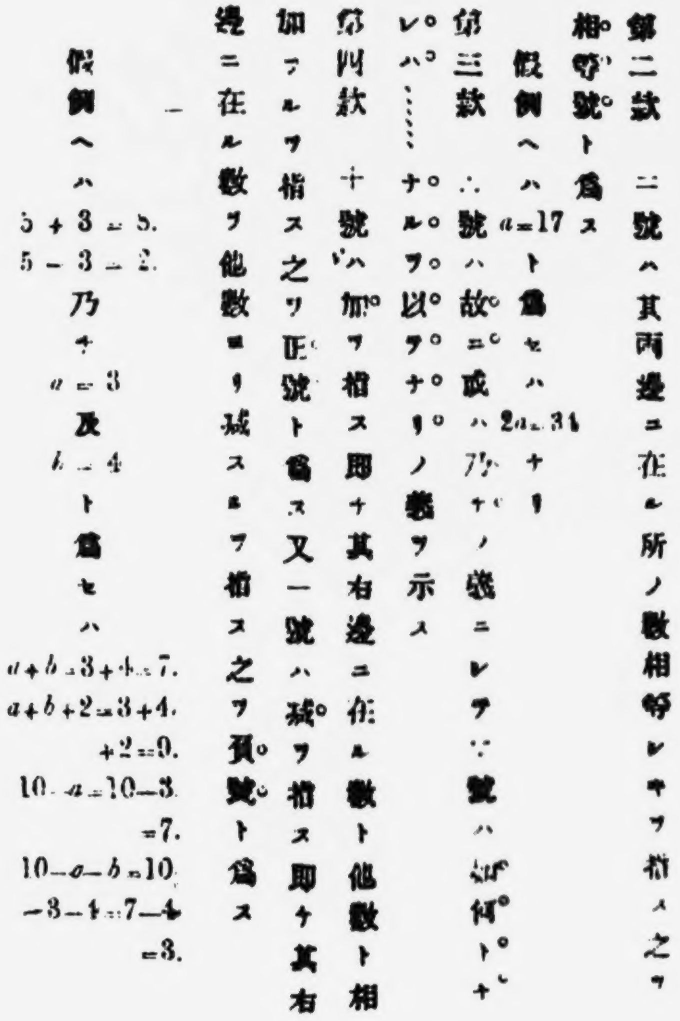
○正負ノ乗術トハ正符或ハ負符ヲ具シタル文字ヲ相乗スルノ法ニシテ乗法正数ナル𪜈ハ乗実常ニ固有ノ符ヲ變ズルコトナク若シ乗法負数ナル𪜈ハ乗実常ニ固有ノ符ヲ變ズルナリ故ニ實法同名ナレバ其積必ズ正数ニシテ異名ナレバ其積必ズ負数トナルナリ乃チ左式ノ如シ

$(+a)\times(+b)=+ab$ （一）

$(+a)\times(-b)=-ab$ （二）

(3) Figure 3 shows a page (12cm x 22cm) from another early publication on Western algebra by unknown, published in 1877. Japanese sentences are written vertically while Western symbolic expressions are inserted among them in horizontal form.

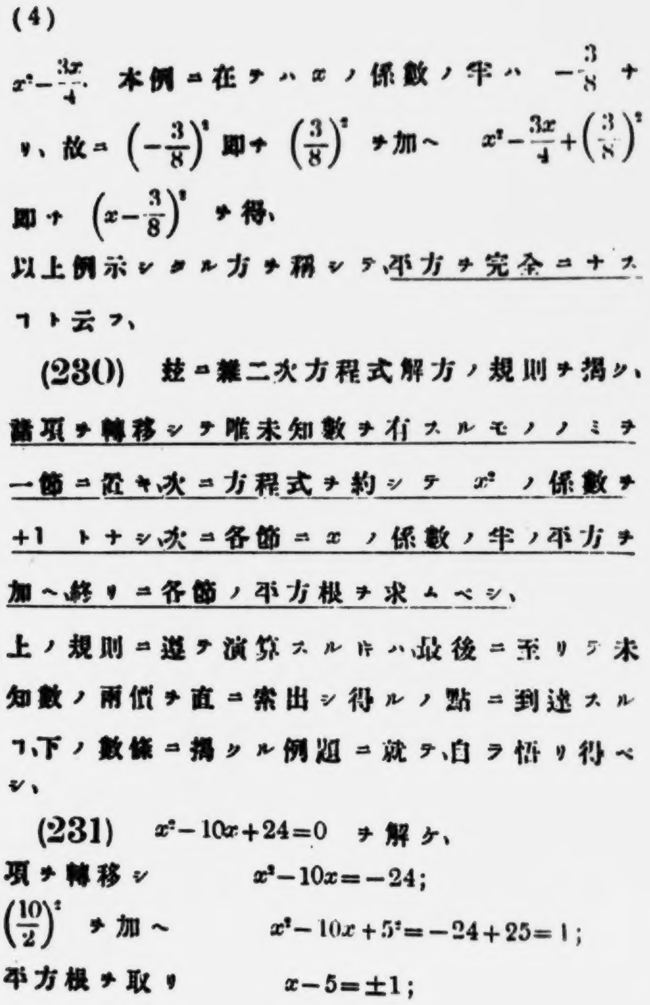
Figure 3



第二款 二號ハ其兩邊ニ在ル所ノ數相等シキヲ指ス之ヲ
相等號ト爲ス
假例ヘハ $a=17$ ト爲セハ $2a=34$ ナリ
第三款 ∴號ハ故ニ或ハ乃チノ義ニシテ∵號ハ如何トナレ
レハ……ナルヲ以テナリノ意ヲ示ス
第四款 ＋號ハ加ヲ指ス即チ其右邊ニ在ル數ト他數ト相
加フルヲ指ス之ヲ正號ト爲ス又－號ハ減ヲ指ス即チ其右
邊ニ在ル數ヲ他數ヨリ減スルヲ指ス之ヲ負號ト爲ス
假例ヘハ $5+3=8.$ $5-3=2.$ 乃チ $a=3$ 及 $b=4$ ト爲セハ $a+b=3+4=7.$ $a+b+2=3+4+2=9.$ $10-a=10-3=7.$ $10-a-b=10-3-4=7-4=3.$

(4) Figure 4 shows a page (12cm x 18cm) from one of the first publications written in the totally horizontal system on algebra by T. Omori and U. Yatabe, published in 1889.

Figure 4



(4)

$x^2-\frac{3x}{4}$. 本例ニ在テハ x ノ係數ノ半ハ $-\frac{3}{8}$ ナリ、故ニ $\left(-\frac{3}{8}\right)^2$ 即チ $\left(\frac{3}{8}\right)^2$ ヲ加ヘ $x^2-\frac{3x}{4}+\left(\frac{3}{8}\right)^2$ 即チ $\left(x-\frac{3}{8}\right)^2$ ヲ得、

以上例示シタル方ヲ稱シテ平方ヲ完全ニナスヿト云フ、

(230) 玆ニ雜二次方程式解方ノ規則ヲ掲ク、

諸項ヲ轉移シテ唯未知數ヲ有スルモノノミヲ一節ニ置キ次ニ方程式ヲ約シテ x^2 ノ係數ヲ+1トナシ次ニ各節ニ x ノ係數ノ半ノ平方ヲ加ヘ終リニ各節ノ平方根ヲ求ムベシ、

上ノ規則ニ遵テ演算スルトキハ最後ニ至リテ未知數ノ兩價ヲ直ニ索出シ得ルノ點ニ到達スルヿ下ノ數條ニ掲グル例題ニ就テ自ラ悟リ得ベシ、

(231) $x^2-10x+24=0$ ヲ解ケ、

項ヲ轉移シ $x^2-10x=-24;$

$\left(\frac{10}{2}\right)^2$ ヲ加ヘ $x^2-10x+5^2=-24+25=1;$

平方根ヲ取リ $x-5=\pm1;$

(Prepared by Nagasaki)

APPENDIX C

Types Of "Application Problems Of Four Operations"

(verbatim translation of the then-used Japanese term)

The following classification is cited from 'Shin Sugaku Ziten' (New Mathematics Cyclopedia), published by Osaka Shoseki in 1979. Most problems had been included in arithmetic teaching before the 2nd World War as a main concern of teachers and pupils, and some still remain in present teaching with somewhat different emphasis.

Those problems were assigned to pupils to apply operations on number and to solve them through an arithmetical method in which use of algebra was not allowed. Usually problems relating to geometrical figures are not included. An example of each type together with its nickname is shown in the following by dividing them into two groups, one being those which appeared in the national textbooks in those days, and the other being those that did not.

1. Those That Appeared In The National Textbooks

During the first half of the 20th century, arithmetic textbooks had been compiled by the Ministry of Education and revised five times during this period. Each version will be referred as T1, T2,..., T6 in chronological order. The years of the first publications are as follows:

T1...1905, T2...1910, T3...1918, T4...1925, T5...1935, T6...1941.

In the following examples, the number of the version, the assigned grade, and the page in the text are shown in the parentheses at the end.

(1) Kiitsu-zan (reducing-to-unit problem)

It takes 50 minutes to copy five pages of a document with 20 letters in a line and 10 lines in a page. How long does it take to copy 20 pages of another document with 25 letters in a line and 12 lines in a page? (T4, Grade 6, p.73 (6))

(2) Ueki-zan (planting-tree problem)

The length of the south side of our playground is 48 meters. We want to put nine flags at equal intervals there. How long is the interval between two successive flags, if we put them at both ends too? (T6, Grade 3, Part 1-p.20, (6))

(3) Sagakuheibun-zan (making-equal-from-different problem)

There are fifteen persons on board a ship and nine persons on board another one. In order to make the number of persons on each ship equal, how many persons should be moved from one to the other? (T6, Grade 3, Part 1-p.22, (13))

(4) Wasa-zan (sum-and-difference problem)

28 decilitres of water is poured into two bottles. The bigger bottle holds 6 decilitres more than the other. How much water is there in each of bottles? (T5, Grade 3, Part 2-p.84, (2))

(5) Ryusui-zan (stream problem)

In the tropics, the east wind blows hard at an altitude of 6000 to 7000 meters. An airplane flew 1100 kilometers in two hours with this wind, and against this wind the plane flew only 150 kilometers in an hour. How fast does this plane fly in still air? How fast does the wind blow? (T6, Grade 6, Part 2-p.73, (7))

(6) Oikake-zan (running-after problem)

Mr. A started from a place at the speed of 4 kilometers an hour. Thirty minutes later, Mr. B started from the same place at the speed of 4.5 kilometers an hour and ran after Mr. A. How long did it take for Mr. B to overtake Mr. A? (T6, Grade 6, Part 1-p.74 (16))

(7) Deai-zan (meeting problem)

Mr. A and Mr. B started at the same time from places 1.05 kilometers apart. They moved towards each other. Mr. A went at the speed of 90 meters a minute and Mr. B at the speed of 85 meters a minute. How long did it take for them to meet? (T4, Grade 6, p.65 (11))

(8) Nenrei-zan (age problem)

The age of a father is 35 years and those of his two sons are 14 and 5 years, respectively. How many years will it take until the sum of two sons' ages is equal to their father's age? (T5, Grade 6, Part 1-p.51, (6))

(9) Tsuru-kame-zan (Cranes-and-tortoises problem)

Altogether, there are 20 cranes and tortoises. The sum of the number of their feet is 52. How many cranes and how many tortoises are there, respectively? (T5, Grade 6, Part 2-p.74, (10))

(10) Soto-zan (rate problem)

21 pages of a Japanese reader has been read but there still remains two thirds the reader. How many pages are there all together? (T2, Grade 6, p.23 (4))

(11) Shigoto-zan (work problem)

Mr. A completes a task in six days, Mr. B in eight days, and Mr. C in twelve days. How many days does it take to complete the task, if they work co-operatively? (T5, Grade 5, Part 2-p.30, (13))

(12) Hojin-zan (square arrangement problem)

Arrange 24 go-stones in a line so as to form a square frame. How many go-stones are there on a side? (T5, Grade 2, Part 2-p.36 (7))

(13) Shuki-zan (period problem)

Streetcars start every 15 minutes from the station and buses every 12 minutes. A streetcar and a bus start at noon. At what time do they start at the same time again? (T5, Grade 6, Part 1-p.23, (3))

Types of problem which were included in each version is shown in the following table.

Table: The appearance of types of problem

(x indicates that the type in the left column is included in that version)

version	T1	T2	T3	T4	T5	T6
type						
(1)	x	x	x	x		
(2)	x		x	x	x	x
(3)	x		x	x	x	x
(4)	x	x	x	x	x	x
(5)					x	x
(6)					x	x
(7)	x	x		x	x	
(8)					x	
(9)					x	x
(10)	x	x	x	x	x	x
(11)	x	x	x	x	x	x
(12)					x	
(13)					x	x

N.B. 1. This table may suggest a change of philosophy to problem-solving from T5 version.

2. Some similar to those mentioned later are omitted here.

2. Other Types That Appeared in Classroom or Preparatory Teaching

Altogether, 23 types are mentioned in the said reference. Those types which are not referred to in the previous section seem to be taught in auxillary fashion or for examination from secondary to tertiary. They are as follows (the number for type continues from the previous one):

(14) Kangen-zan (working-backwards problem)

After spending a half of the money, he had received 2000 Yen. Then he spent 1000 Yen more money than a half of the money, and found 2500 Yen in hand. How much had he at first?

(15) Teiwa-zan (constant-sum problem)

There are 8.6 dl sauce in bottle A and 3.4 dl in bottle B. How much must be poured from A to B in order to make the volume of sauce in B double of that in A?

(16) Teisa-zan (constant-difference problem)

There were 8.6 dl sauce in bottle A and 3.4 dl sauce in bottle B. When the same volume of sauce was poured into both bottles, the volume in A became twice that in B. How much sauce was poured into both?

(17) Baisu-zan (multiple problem)

There were all together 12 dl sauce in bottles A and B. After 1.4 dl of A was used and 0.2 dl was added to B, the volume in A became as twice as that in B. At first, how much sauce was there in A and B, respectively?

(18) Baisu-henka-zan (multiple-change problem)

Mr. A has 210 postal cards and Mr. B 190 postal cards. If Mr. A uses 16 cards a month and Mr. B 14 cards a month, how many months does it take until their remainders become equal in number?

(19) Kafusoku-zan (Excess-and-defect problem)

A heap of oranges is to be distributed to each child in a group. If each child receives three oranges, ten oranges will be left. If each child receives five oranges, four more oranges will be needed. How many oranges are there? How many children are there?

(20) Tsuka-zan (passing-through problem)

The train is 200m long and goes at 100km/h. It passes through a tunnel whose length is 4km. How long does it take from the time the head of the train comes to the tunnel to the time the tail of the train goes out the tunnel?

(21) Heikin-zan (average problem)

There are ten bottles containing sauce. The average volume of sauce among ten bottles is 8.55dl. Only one of them contains 7.5dl and the others contain 9dl or 8.5dl. How many bottles of 9dl and 8.5dl are there?

(22) Shokyo-zan (elimination problem)

There are two articles in a shop, A and B. The price for three A's is equal to the price for four B's. The price of one A and one B is 2100 Yen. How much does each of A and B cost, respectively?

(23) Tokei-zan (clock problem)

At what time do the long hand and the short hand of the clock overlap each other between 7 o'clock and 8 o'clock?

(Prepared by Nagasaki)

Appendix D

Examples Of Strategies Or Tactics For Problem Solving In Mathematics Advocated By Authors Of Popular Crambooks For Secondary School Students

Because of a severe competition in the entrance examination to famous universities, many crambooks were and are published to help those who are preparing for the examination. Authors give advice on how to study as well as a kind explanation of the theory in question and solution of related problems. Two examples are given below:

1. Those by R. Fujimori

Fujimori worked actively during about 1910-1940's and published a series of crambooks entitled "How to study, think, and solve in the subject x," in which he advocated the following. His work was succeeded by his son, Y. Fujimori (translated from a volume in 'Basic analysis' by Y. Fujimori, published in 1953).

(1) The systematic way of studying mathematics:

to select vital matters in the subject and arrange them in order,
to understand and memorize them,
to select, arrange in order, and study main problems from which you can learn how to solve,
to master the methods by which you can solve similar or new problems,
to foster background to transform or construct new problems,
and thus, to attain a stage of creation and invention.

(2) Examples of tactics:

be not reluctant to repeat.
two major weapons to save thinking are
  saving thinking by use of algebraic expression
  saving thinking by use of geometric diagram.
form an attitude to ask in yourself in the following order, when you are given an algebraic expression:
  1. what variables are involved?

2. of what degree is each?

2. Those by J. Hashimoto

Hashimoto was a mathematician and a successor of K. Hoshino who had invented the Chart System as a style of crambook during about 1930's-1940's, and he actively worked in this respect during about 1940's-1970's. His strategy is as follows:

(1) A proper attitude to be given problems is needed for solving them successfully. That is:

to understand the meaning of the problem, to have a plan to solve it, to write a paper to answer it, and to check the paper.

Among these four, it is the second step which largely influences success or failure. The general principle for this second step is:

(i) to connect what is required with what are given,

(ii) when a direct connection is difficult to find, consider an intermediate means between them,

(iii) when a front gate cannot do, try a rear gate,

(iv) consider whether you forgot some of conditions or not.

(2) Tactics for planning solution in algebraic problems.

(i) Consider what should be represented by variables.

(ii) Consider how should the given variables be regarded.

(iii) Consider how to express the essence of the problems in terms of variables.

(iv) Consider how to transform the expressions thus obtained.

(3) Further examples of tactics.

(i) Before proceeding, arrange a given expression in a decreasing order of terms.

(ii) The first principle of a factorization

At first, arrange them with respect to the variable of the lowest dagree.

(iii) $\sqrt{(\text{fear})^2}$: if $a \geq 0$, then $\sqrt{a^2} = a$

if $a \leq 0$, then $\sqrt{a^2} = -a$

(Prepared by Nagasaki)

Discussion of Professor Shimada's Paper

Becker: We thank Professor Shimada for providing an historical background for Japanese mathematics education and the influence of Western mathematics on Japanese mathematics education. Probably many of us Americans are not too familiar with this background. We now have approximately half an hour for discussion. Let us open it up for discussion and, perhaps, I can start it off.

Professor Shimada made reference to the role of entrance examinations in Japan, and he mentioned that there are some not good characteristics of them. But yet, we know from data of the Second International Mathematics Study that Japanese students perform very well. Do the entrance examinations have some positive effects as well as some negative effects?

Shimada: Perhaps for an evaluation such as the IEA it may have had positive effects. It made students strong in solving routine items on the test, and made more items routine work for Japanese students than for students in other countries.

But when I was engaged in the first IEA Mathematics Study, I found that Japanese students' view of mathematics was very much static. That is, they saw mathematics in a very static way and not in a dynamic and functional way as it should be. This means a further development of their mathematical ability may be very limited.

This finding stimulated me and others to start studies on how to develop an open-minded and dynamic attitude towards mathematics on the part of students.

Becker: I should have adhered to the rule that we need to establish during the discussion by first stating my name, then asking my question in English, and then letting the translators translate it into Japanese. I'm sorry about that. So the next person will please do it right.

Travers: I found your comments very helpful, Professor Shimada, in providing a background in helping to identify some cultural aspects of problem solving. I would like to try to explore one aspect of differences in mathematics education between Japan and the United States to which you allude in your paper. Perhaps you could expand on it a bit. It has to do with the role of computation in contrast to problem solving. From data we have seen, for example, it looks as though in the United States, in the early grades, there is a great deal of concern with bringing children to a satisfactory level of computational ability in arithmetic. Even at the junior high school level there is almost a domination of the curriculum by computational aspects of mathematics. However, in the Japanese data, we see that arithmetic is essentially dealt with by the age of eleven or twelve and that students have reached very impressive levels of mastery. Can you identify factors that would account for this difference? I'll raise one possibility, though I'm sure there are many. Does it have anything to do with, from my point of view, the integral role of the abacus as an important part of the culture?

Shimada: Usually the teaching of the abacus in school is not so much emphasized in Japan. It is included in the syllabus, but not too much time is assigned to its teaching and most students carry out their calculation without an abacus unless specifically requested to do so; rather, they use the pencil and paper algorithms. But in this case, it is a good point for Japanese students to have a clear decimal numeration system in the Japanese language which is very sensitive to the place value. Being sensitive to place value is a kind of inheritance from the use of the abacus (Soroban). And usually if students succeed in mastering the basic number facts (i.e., computation involving two one-digit numbers) then they do not have so much difficulty in learning further computation with numbers having many digits. (Japanese students work hard on exercises on computation.)

The difference between our two countries may be due to two factors. One is the numeration system to which I just made reference, and the other may be the expectation of students by parents and society. Parents expect that their children can do such things, teachers also have the same expectation. So students automatically try to live up to these expectations. Even in the case of parents who think of themselves as weak in computation, they commonly encourage their children to work hard so as to become strong in computation. Perhaps, this parental attitude may be somewhat different from that in the U.S.

Travers: One other point. Would you say that there is a great emphasis on mental computation as opposed to, say, using pencil and paper?

Shimada: Mental computation was once very much emphasized in the elementary school. But after World War II, this emphasis diminished. During the war in 1935-45, the three ways of computation (paper-pencil, mental, and with Soroban) were given equal weight in teaching. But after the War, this policy was criticized as demanding too much of students, especially by the officers in charge of education in the General Headquarters of the Occupation Force. Since then the emphasis for mental computation has been decreased. Today the addition of a two-digit number and a two-digit number and multiplication of a two-digit number by a one-digit number may be required to be done mentally.

Travis: Professor Shimada, I found your paper and your perspective on the historical development of mathematics education in Japan very interesting. I have a couple of questions that are related, so I'll give them both at the same time. Is the use of the calculator encouraged in the elementary grades? And what role does the calculator play in problem solving in Japan?

Shimada: Usually the calculator is not widely used in elementary school teaching. But one of my colleagues studied its potential to enrich mathematical learning in the elementary school. To enrich means to use it in problem solving. If the use of the calculator is allowed in classrooms, then students would be able to concentrate on the way to find the solution and expressing it in a horizontal form of expression. This approach would allow students to cover more exercises on problem solving in a given time than otherwise, because computation is done on the calculator. My colleague reported on his research in the Regional Conference on Mathematical Education held in Tokyo a few years ago.

Sawada: In the results of IEA study, concerning the use of the calculator in Japan, almost none of the teachers used calculators in their mathematics classrooms, in both the junior and senior secondary schools.

Becker: Professor Shimada, when you made reference to doing mental computation, multiplying a two-digit number by a one-digit number and adding two two-digit numbers, do you mean mental computation using the Soroban or without the use of a Soroban?

Shimada: As far as I know, the algorithm for mental computation is somewhat different from that used on the Soroban. But those who are competent in using the Soroban may do it in a "mental" Soroban - they use an image of the Soroban mentally.

But in classroom teaching, mental computation is carried out in another way. For example, to compute 25 + 37, pupils process it as follows:

1) twenty five and thirty is fifty five.
2) fifty five and seven is sixty two.

We call this method "from-head-to-tail," or "from highest place to unit place" method. Paper-pencil computation is carried out in an opposite direction, that is to say, in "from tail-to-head" or "from unit place to highest place" method. Numbers are spoken and read from-head-to-tail, and so the from-head-to-tail method is more natural than the reversed way.

34 x 4 is done in the following way. Thirty by four is one hundred and twenty. Four by four is sixteen. So it is one hundred and thirty six. Also in this case, the process starts from the highest place.

In using the Soroban, the same procedure is adopted wholey in addition and partially in multiplication, while attention to place value must be paid only to its relative value to the next one, but not to its absolute value, as in mental computation. Furthermore, the use of the Soroban requires mastering the decomposition of 5, together with that of 10.

So, the three methods of computation - paper-pencil, mental, and soroban - are different from each other in the way of processing information. This is the reason why mastery of all three methods was thought too demanding for average students and emphasis on mental computation has consequently decreased since 1945.

Becker: How about an example of multiplying a two-digit number by a two-digit number mentally?

Shimada: This is beyond the requirements, and is not included in our programs.

Clarkson: College students in the United States must relearn computation and problem solving when they are adults because they have either forgotten or they never really understood it to begin with. I'd like to ask two questions: first of all, do you see the same problem in Japan? Secondly, what is the role of review from year-to-year in the Japanese mathematics curriculum?

Shimada: For the first question: It depends on the faculty of the university. In most Japanese universities, students must decide what or what kind of faculty (program of study) they want to enter before they apply for the entrance examination. In the faculties of literature, humanities, or others you may find those students who are very incompetent in mathematical problem solving or computation. But in the faculties of science, medicine, engineering, economics, or others you may not, because students must pass some kind of mathematics examination.

For the second question: in the official curriculum no reference is made to review of the previous work. It is left completely to teachers' judgment. Usually teachers assign some amount of homework regularly for the review. Usually they emphasize review rather than preview. According to the results of the First IEA Mathematics Study, the amount of homework in terms of needed time for Japanese students was larger than those for other countries.

Becker: Unfortunately, our time is up. This concludes the discussion of Professor Shimada's paper. Before we close the opening session, however, Professor Nohda would like to make a presentation.

Nohda: In appreciation for inviting the Japanese participants to this Seminar, we would like to present a small gift to each of our American colleagues. I will give Professor Becker the gifts, and Professor Becker can distribute them to each of you.

Becker: Professor Nohda, we thank you and all the members of the Japanese delegation for these nice gifts, which everyone has now discovered are calculators. This is a very nice gesture and we appreciate it very much. For each of us, the calculator will be very useful and a constant reminder of these days together with Japanese colleagues.

Professor Silver's Paper

Miwa: We are now starting the next part of Session 1. The speaker is Professor Edward Silver. He is very well known in the mathematics education community as is his excellent work in mathematics education. It is not necessary to say more so I present Professor Silver.

Silver: I am grateful for that introduction and for the gift of the calculator. It is a great personal honor for me to be the opening speaker for the U.S. delegation. Now I will go to my paper.

RESEARCH ON MATHEMATICAL PROBLEM SOLVING
IN THE UNITED STATES OF AMERICA:
SOME RECENT TRENDS

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This paper discusses the current state of research on mathematical problem solving in the United States, focussing particularly on some recent trends. Since this paper will necessarily be brief, I would suggest that the reader interested in more information could also consult any of a number of books that have been recently published in the United States, each of which deals more comprehensively with current theoretical and research perspectives on mathematical problem solving (e.g., Davis, 1984; Schoenfeld, 1985; Silver, 1985).

THE CURRENT STATE: BACKGROUND

It has been said that "problem solving is the 'new mathematics' of the 1980s." In much the same way that the "new math" captured the attention of a cross-section of American society, including not only the educational community but also the wider society as well, "problem solving" has become widely accepted as a topic of importance for mathematics education.

There are many indicators of widespread general interest in mathematical problem solving. Various professional societies have proclaimed the importance of problem solving as one of the basic goals for mathematics education. For example, the National Council of Supervisors of Mathematics (1978), in its position paper on basic skills, declared that "learning to solve problems is the principal reason for studying mathematics." Two years later, the National Council of Teachers of Mathematics (1980) declared that problem solving should be the "focus of school mathematics in the 1980s."

Problem solving is a nearly ubiquitous topic on the agenda of local, state, regional, and national conferences of mathematics teachers. The popular journals for mathematics teachers, such as the Arithmetic Teacher and the Mathematics Teacher, regularly feature articles, and sometimes entire issues, on problem solving. Conferences of researchers, such as the annual meeting of the American Educational Research Association and the Research Pre-Session to the annual meeting of the National Council of Teachers of Mathematics, have also devoted many sessions to problem solving on a regular basis throughout the 1980s.

The interest in problem solving is also evident in the textbooks that are currently being produced in the United States, especially at the elementary school level. Problem solving is now highlighted in both the student's text and teacher's edition. Moreover, the textbook authors' explicit attention to the development of problem-solving activities for the students is often featured in the advertising that is done for the textbook series. It is true that there are some textbook authors (e.g., Saxon, 1985) who have been very successful without including problem solving as a focus of their textbooks, but the general trend is clearly towards the widespread inclusion of problem-solving activities throughout an entire elementary school series and, to a somewhat lesser extent, also in the core courses at the secondary school level (e.g., General Mathematics, Elementary Algebra, Plane Geometry).

Problem solving has also become a dominant topic in textbooks written for prospective teachers of mathematics, especially for those preparing to teach at the elementary school level. For prospective secondary school teachers, many universities have designed special courses dealing with problem solving. Furthermore, the emphasis on problem solving is not limited to preservice education; it is a common topic of inservice institutes as well.

The interest in problem solving as a goal in mathematics education also spreads beyond narrow professional interests. Several governmental commissions studying the current state of education in the United States have issued pronouncements that include calls for greater attention to problem solving. For example, the National Science Board Commission on Precollege Education in Mathematics, Science, and Technology asserted in its recent report (1983) that:

> "Analysis of current student performance in mathematics - particularly the use of mathematical skills in unfamiliar areas - indicates that they are learning to be technicians but not problem solvers. Opportunities should be provided for the application of arithmetic and mathematics in a variety of areas - in the natural and social sciences, in consumer-related experiences and in other real-life situations where analysis through mathematics is possible." (pp. 42-43)

Although there are undoubtedly many in our society for whom mathematics is synonymous with arithmetic computation and algebraic symbolism, it is clear that awareness of the importance of problem-solving competence as a higher-level goal of mathematics education is now evident to a great extent.

The above treatment of the current state of interest in mathematical problem solving in the United States was quite brief. Many of these issues are dealt with more extensively in other papers prepared for this conference, especially in the excellent paper by Wilson.

THE CURRENT STATE: MULTIPLE MEANINGS

Given the widespread interest in problem solving, many suggestions have been made about how problem solving might be incorporated as a fundamental goal of mathematics education. The popular pedagogical literature is filled with articles suggesting "innovative" approaches to the teaching of problem solving. The careful reader of this literature will note that there is no general agreement about the meaning of the term "problem solving." In fact, it is not at all unusual to find that two authors may propose apparently inconsistent approaches because they have very different conceptions about the nature of mathematical problem solving.

For some authors, the domain of mathematical problem solving consists solely of standard textbook story problems (e.g., "John has 3 cartons of soda, each of which contains 6 bottles. How many bottles does he have?"). Other authors call standard story problems "exercises," because of their routine nature, and reserve the term "problem" for less routine tasks (e.g., "If there are 10 people at a party and each person shakes hands once with everyone else at the party, how many handshakes will there be?"). This latter nonroutine problem may require students to do something other than simply apply a well-learned algorithm or procedure. Of course, some authors are willing to include both kinds of tasks as problems in their definitions.

There are other distinctions and differences of opinion that are fairly common in the literature. For example, some authors may treat logical puzzles and games as mathematical problems, whereas other authors would exclude these tasks from their definitions, perhaps because they do not involve any numerical calculation. And debates over "applied" problems, whose content is primarily in some field other than mathematics (e.g., physics) or in "real-life" situations (e.g., How much wallpaper is needed to decorate this room?), and their relation to standard textbook problems are not at all uncommon.

As one might expect, confusion about the meaning of the term "problem solving" is closely related to confusion about the nature of problem solving as a goal of mathematics education. For some, the goal is to help children learn to solve standard textbook story problems successfully; for others, the goal is to endow students with powerful general problem-solving strategies that could be applied across many mathematics problems. I think it is correct to say that, at this time of widespread interest in the United States, there is general agreement neither about the general meaning of the term "problem solving" nor of its meaning as a goal of mathematics education. What is somewhat clearer, however, is the set of data that has been obtained from national and state assessments concerning the problem-solving competence of American students.

THE CURRENT STATE: ASSESSMENT RESULTS

Despite the generally accepted importance of problem solving, it is clear from the results of the National Assessment of Educational Progress (NAEP) and most state mathematics assessments (e.g., California Assessment Program, 1985) that many students are not always capable of solving relatively straightforward mathematics problems, and most students fail to solve somewhat complex problems. On virtually all mathematics assessments, problem solving is the area in which performance is the poorest. The problem-solving success that students generally do have is in the area of solving routine one-step word problems such as those found in typical textbooks. On the other hand, there is a marked decline in student performance on problems that require some analysis or nonstandard application of knowledge or skills.

In general, the assessment results suggest that the majority of students at all age levels have difficulty with any nonroutine problem that requires some analysis or thinking. Students do not carefully analyze the problems they are asked to solve, and they have not learned basic problem-solving skills. The errors made on many of the problems suggest that students generally attempt to apply mechanically some mathematical calculation to whatever numbers are given in a problem, without regard for the relationship of either the given numbers or the resulting answers to the problem situation. This apparent lack of student understanding in problem solving was evident in performance on several NAEP exercises, for which students gave answers suggesting that they had routinely performed correct calculations without analyzing the problems sufficiently to determine the required information.

Since the NAEP data have been discussed extensively elsewhere (e.g., Carpenter, Lindquist, Matthews, & Silver, 1983; National Assessment of Educational Progress, 1983), I will not repeat that discussion here. Therefore, we will consider only one well-publicized NAEP example given to 13-year-olds:

> "An army bus holds 36 soldiers. If 1,128 soldiers are being bused to their training sight, how many buses are needed?"

About 70% of the students performed the correct calculation, but about 29% gave the exact quotient (including the remainder) and another 18% ignored the remainder. These answers may reveal a failure to understand the problem situation and the nature of the unknown. Those who responded with the exact quotient disregarded the need for a whole number of buses, and those whose response ignored the remainder failed to provide transportation for all the soldiers. A similar problem, with similar performance results, has been given to sixth- and eighth-grade students on the California Assessment Program. Silver (1986) has provided an extensive discussion of this item and related research into the causes of the poor student performance.

These results, together with the findings on problems concerning missing or extraneous data, suggest that even when students are successful, they may not understand the problems they solve. Since most of the routine verbal problems presented in elementary textbooks can be solved by mechanically applying computational algorithms, there is no need to understand the problem situation; nor is there any reason to consider why a particular computation is appropriate, or whether an answer is reasonable. However, when students are given nonroutine problems in which these and other considerations are important, their performance declines considerably.

THE CURRENT STATE: RESEARCH TRENDS

Almost 20 years ago, Kilpatrick (1969) reviewed the literature on mathematical problem solving and concluded that "problem solving is not being systematically investigated by mathematics educators" (p. 523). In the past two decades, the situation has changed dramatically. During this time, there has been a considerable amount of research dealing with the nature of mathematical problem-solving performance. Much of the research has been conducted by cognitive psychologists, seeking to develop or validate theories of human learning and problem solving, and by mathematics educators, seeking to understand the nature of the interaction between students and the mathematical subject matter that they study. Moreover, the work tends to be much more systematic than that reviewed by Kilpatrick in 1969.

The remainder of this paper consists of a discussion of other comparisons between the state of research on mathematical problem solving, as portrayed in Kilpatrick's review, and the current state as I view it. In particular, some current theoretical emphases, methodological approaches, and research themes will be discussed.

THEORETICAL EMPHASES

In his review, Kilpatrick (1969) decried the fact that few studies had an explicit theoretical rationale or built on previous research, but he noted signs of increased interest on the part of mathematics educators in psychological theory and research related to higher-order cognitive processes. Kilpatrick's perception of an emerging trend was apparently correct, for the current situation is quite different from the one he reviewed in 1969. The influence of modern cognitive psychology on current problem-solving research has been substantial.

In the United States, most current research on mathematical problem solving is based on cognitive psychology. In this paper, we can only consider a very brief account of modern cognitive theory. More complete treatments can be found in Frederickson (1985), Schoenfeld (1985), and Silver (1987).

Modern cognitive theories typically rest on assumptions about memory and information processing. Memory is typically conceived of as consisting of a short-term or working memory (WM) and a long-term memory (LTM). Working memory contains the information that is actively being used at any given time. Information can be stored in LTM only after being processed in WM, and it can be used in thinking only after being retrieved from LTM and placed in WM. Thus, the cognitive activity called "information processing" consists of controlling the flow of information into and out of WM by processes such as receiving information from the sensory buffer and retrieving information from LTM; recognizing, comparing, and manipulating symbols in WM; and storing information in

LTM. As the term "information processing" implies, cognitive theories have been heavily influenced by computer metaphors.

According to cognitive theory, the limitations that are imposed by the limitations on the capacity of WM (generally agreed to be 7 ± 2 items) are substantial and have dramatic consequences. Much of cognitive research consists of the examination of consequences of the hypothesized limitations on human information-processing capacity, and on strategies for overcoming the limitations, such as "chunking" and automatic processing.

A second contemporary theoretical thrust is the assumption that human learning is largely a "constructive" process. One of the fundamental assumptions of recent research on mathematics learning and problem solving is that new knowledge is in large part constructed by the learner. According to this view, learners do not simply add new information to their store of knowledge; instead they integrate new information into already established knowledge structures and build new relationships among those structures. This process of building new relationships is essential to learning. The recent versions of constructivism are largely compatible with earlier versions, such as Piaget's theories of human learning, although the terminology is somewhat different.

One of the consequences of a constructivist view of learning and problem solving is that systematic errors or "bugs" can occur. Another consequence is that constructed knowledge, especially in the form of "misconceptions" about phenomena, can be quite resistant to instruction. Each of these constructivist consequences has been the subject of research attention in recent years. For example, Brown and his colleagues (Brown & Burton, 1978; Brown & van Lehn, 1980) have extensively studied the "bugs" that arise in children's learning of the subtraction algorithm in elementary school. Mathematical misconceptions, especially in statistics and probability, have also been studied (c.f., Shaughnessy, 1985).

METHODOLOGICAL APPROACHES

In 1969, Kilpatrick noted that most studies of problem solving were either "one-shot comparisons of ill-defined 'methods'" or "laboratory studies of arbitrary, highly artificial problems." Given the limited state of our knowledge about mathematical problem solving, he argued that researchers might be well-advised to consider clinical studies of individual subjects. His advice was apparently heeded, because contemporary research has heavily emphasized clinical approaches and, to a somewhat lesser extent, case studies of individuals. Moreover, "one-shot" comparison studies have completely disappeared, and highly artificial problems have been almost completely eliminated from the serious research literature on mathematical problem solving.

Recent research has typically involved problem tasks that are drawn from actual textbooks, or realistic problems from students' lives, or problems that are nonstandard but appropriately related to the mathematics that students have studied. Explicit attention is given to the knowledge that a person would need to know in order to solve the problem, task and the processes used by the solver.

The most popular technique for studying the processes used in mathematical problem solving has been the "talk aloud" clinical interview. This technique, pioneered by Gestalt psychologists in the 1930s and 1940s (e.g., Duncker, 1945), has been widely used to study both the cognitive and metacognitive aspects of problem-solving episodes.

Another current approach, popularized by some cognitive psychologists has involved the development of computer simulations of problem solving. For example, Larkin (1980) developed a program called ABLE, which learned to solve increasingly complex (though fairly elementary) physics problems by using its problem-solving experiences to augment its store of knowledge. ABLE's knowledge consisted of "productions" (condition-action pairs). It utilized this knowledge to solve problems by matching the condition part of a production with the contents of its working memory. Beginning with a version of the program called "Barely ABLE," it learned to improve its performance by acquiring new productions as a result of its problem-solving experiences with representative problems. Larkin's simulations of novices (Barely ABLE) and experts (ABLE) were quite similar in many respects to the behavior of human problem solvers.

A somewhat different computer-based approach has involved the construction of "intelligent tutors" (c.f., Anderson, 1982) that provide problem-solving instruction. The tutorial interaction between the learner and the computer tutor provides a rich data source for research into the nature of and requirements for mathematical problem solving.

CURRENT THEMES

Kilpatrick (1969) organized his discussion around five categories: problem-solving ability, problem-solving tasks, problem-solving processes, instructional programs, and teacher influences. These categories vary in the extent to which they are addressed in current research. For example, classical research on problem-solving tasks and the characteristics that contribute to task difficulty - with an emphasis on linear regression models for predicting task difficulty - has given way to a detailed consideration of the ways in which task characteristics interact with individual cognitive functioning. Similarly, in the study of problem-solving ability, the widespread use of factor analytic approaches and the treatment of problem-solving ability as a (nearly) unitary phenomenon have been replaced by clinical studies of the nature of problem-solving expertise, with a heavy emphasis on the study of processes.

In this portion of the paper, we will consider two major areas of current research and the themes associated with each area. The first is problem-solving expertise and the second is problem-solving instruction.

PROBLEM-SOLVING EXPERTISE

The discussion of problem-solving expertise is organized around five themes which have been dominant in much of the recent cognitive research. The themes of pattern recognition, representation, understanding, memory schemas, and meta-processes are each discussed in turn.

Pattern Recognition

In a classic study, deGroot (1965) asked chess experts (grandmasters and masters) and novices (ordinary chess players) to reproduce the position of the pieces on a chess board. The pieces were either arranged in a mid-game position or randomly arranged on the board. For the mid-game positions, the experts were able to reproduce the positions of the 20 or 25 pieces almost without error, while ordinary players could place only a half-dozen pieces correctly. For the random arrangements, only about 6 pieces were correctly placed both by masters and ordinary players.

Chess experts' ability to recognize patterns of related pieces on the board almost instantaneously and to use these patterns rather than the positions of individual pieces in processing information, is consistent with the findings of other research on expert knowledge in complex task domains. Skilled medical diagnosis may also involve pattern-recognition skills. In one study (Norman, Jacoby, Feightner, & Campbell, 1979), written case histories were presented to doctors with varying levels of experience and training (practicing physicians, third-year residents, first-year residents, and second-year medical students). The subjects were asked to read each case history and then write out as much of it as they could remember. For the histories based on common diseases, experienced physicians recalled the most details, followed by the other groups in descending order of experience and training. For the histories that contained findings not suggestive of any disease, there was little difference among the groups. Skilled diagnosis apparently involves perceiving patterns of signs and symptoms that correspond to disease entities.

Davis (1984) has argued that much of mathematical expertise is also captured in the recognition of patterns. Davis and his colleagues (Davis, Jockusch, & McKnight, 1978; Davis & McKnight, 1979) have identified a number of characteristics of skilled algebraic task performance that are pattern-based. Davis (1984) has also supplied examples from calculus problem solving, in which appropriate problem-solving techniques are "triggered" by certain perceptual features of the problem task.

Although pattern recognition, pattern finding, pattern generation, pattern extension, and other pattern-related processes appear to be quite important components of mathematical activity, the processes have not been extensively studied in mathematical contexts. The careful study of these processes might be one interesting component of a cross-cultural mathematics education research agenda.

Representation

Many current theories describe problem solving as a process of building successively richer and more refined problem representations. The solver begins with an initial representation, then gradually elaborates and refines it until he or she obtains a final problem representation that is adequate for the solution. Lesh (1985) proposes an alternative view, in which the problem solver builds and then abandons unstable representational models of the problem until reaching a stable model. In either view, problem representations are central to the problem-solving process. Consequently, investigators have looked at the representations formed by successful problem solvers to see if they are in any way different from those formed by less successful solvers.

Larkin (1980) studied the problem solving of experts and novices in the area of physics. She noted that experts frequently formed qualitative representations of the problems before attempting any quantitative analyses. They often mentally replaced the original problem with an abstracted version that retained its general structure and features and then used this idealized representation as a guide in solving the original problem. In contrast to the expert behavior, novices usually initiated quantitative analyses even when their problem representations were inadequate and insufficiently constrained to suggest the correct procedures.

Clement (1983) has noted that experts in mathematics and some scientific domains engage in metaphorical processes as they construct problem representations: that is, they look for analogies between the problem at hand and other situations with which they are familiar, and they use these analogies to suggest possible representations of the problem to be solved. Clement notes that these analogies often take the form of mental images.

Representations play an essential role in the problem solver's understanding of the problem and in the recognition of relationships with other problems. Students who build similar representations for mathematically related problems are far more likely to notice their similarity and to use the relationship about one problem in solving the other. Conversely, the similarity betwen mathematics problems with isomorphic representations can go unnoticed if the solver does not represent the two problems in similar ways.

The importance of problem representations is evident even in work with very young children. Researchers who have studied young children's solutions of addition and subtraction problems are virtually unanimous in their agreement that the major factor in attaining problem-solving skill in that domain is the development of problem representation skills (Briars & Larkin, 1984; Riley, Greeno, & Heller, 1983).

Representations play a fundamental role in the current theoretical formulations of human problem solving. From a cross-cultural perspective, it might be interesting to study the representations utilized by students from different cultural and educational systems as they try to solve a set of common mathematics problems. Such a study might give us some fundamental insights regarding the extent to which representation systems for mathematics problems are inherent in the problems or are a function of experiential and cultural factors.

Understanding

As Brownell (1942) observed, "Skill in problem solving is partly a matter of technique and partly a matter of meanings and understanding" (p. 439). The role of understanding in problem solving has also been emphasized by Polya (1957), for whom the first phase of problem solving was "understanding the problem."

In recent years, a growing body of research on problem solving has focused on children's solutions of simple addition and subtraction word problems (Carpenter, Moser, & Romberg, 1982; Riley et.al. 1983). Carpenter (1985) provides an excellent summary of this research and its relationship to other research on mathematical problem solving.

One of the most interesting aspects of the work on young children's problem solving is the finding that, even before they had received formal instruction in arithmetic, almost all children were able to solve some simple word problems using appropriate modeling and counting procedures (Carpenter, Hiebert, & Moser, 1979). However, after receiving formal instruction, students no longer exhibited a rich variety of approaches and were less apt to develop a solution that modeled the given problem. After instruction, students were very likely to write number sentences to represent the problem, but they were generally unable to relate the number sentence to other solutions that could be formed by modeling or counting (DeCorte & Verschaffel, 1983). In other words, the child's mathematical representation of the problem was unrelated to the child's semantic representation of the problem.

The analysis of successful children's solution processes suggests that they attend to the semantics of the problem situation: that is, they succeed because they have an adequate understanding of the problem situation. As Carpenter (1985) has noted:

> Even before they have received instruction in formal arithmetic, almost all children exhibit reasonably sophisticated and appropriate problem-solving skills in solving simple word problems. They attend to the content of the problem; they model the problem; they invent more efficient procedures for computing the answer. Given the limits of their mathematical knowledge, this performance is remarkable. (p. 37)

This finding is consistent with other research on problem solving (e.g., Paige & Simon, 1966). It is striking that many of these same children will no longer attend to the semantics of the problem after receiving mathematics instruction in school; they prefer instead to choose an arithmetical operation on the basis of the problem's surface features.

Successful problem solvers are able to build sophisticated representations for more complex problems, and they are able to connect their mathematical representations with their understanding of the problem situation. Unfortunately, many students never develop this ability, and their difficulties with problem representation and problem understanding increase as new operations and content (e.g., multiplication, fractions) are added to the curriculum. The end result of this failure is too often a student whose approach to problem solving is simply the mechanical application of arithmetic operations on the basis of the most superficial reading of the problem, as was noted in the earlier discussion of assessment results. As one fifth grader suggested, when asked how he solved word problems, "You look at all the numbers in the problem. Then you go to the next-to-last period and read on from there. That tells you what to do" (Lester & Garofalo, 1982, p. 10).

The importance and role of understanding in problem solving is likely to be universal, yet the particular kinds of understandings or misunderstandings associated with problematic situations might well vary across cultures. This issue might also form part of a cross-cultural research agenda.

Memory Schemas

Simon (1980) has noted that "research on cognitive skills has taught us...that there is no such thing as expertness without knowledge - extensive and accessible knowledge." Polya (1973) observed earlier that "a well-stocked and well-organized body of knowledge is an asset to the problem solver. Good organization which renders the knowledge readily available may be even more important than the extent of the knowledge." As we have seen, information stored in LTM plays an important role in problem-solving theories. Since the efficient retrieval of information may depend on the way that information is organized, in LTM differences in problem-solving success may be partly attributable to differences in problem solvers' knowledge organization.

The notion of a memory schema (a cluster of knowledge that describes the typical properties of the concept it represents) has recently helped explain many aspects of human knowledge organization and recall, especially in the area of prose text learning. In the past five years or so, a considerable amount of research has been generated on the influence and use of schemata. A schema is usually described as representing a prototypical abstraction of a complex and frequently encountered concept or phenomenon (e.g., Thorndyke & Yekovich, 1980), and it is usually derived from past experience with numerous exemplars of the concept involved. Schemata have been associated with not only interpreting and encoding incoming information, but also with recalling previously processed information (Thorndyke and Hayes-Roth, 1979). They can influence the efficiency with which information is recalled from memory (e.g., Mandler and Johnson, 1977). Furthermore, schemata may account for inferences made using incomplete information (e.g., Bransford, Barclay, and Franks, 1972).

Examples of the role played by schemas in successful problem solving come from studies that have examined the differences between relative experts and relative novices in physics (Chi, Feltovich & Glaser, 1981; Chi, Glaser, & Rees, 1981) and mathematics (Schoenfeld & Herrmann, 1982) and between successful and less successful problem solvers in mathematics (Silver, 1979). In the expert/novice comparisons, both groups were asked to categorize problems according to similarities in solution methods. The novices tended to sort on the basis of surface features, whereas the experts categorized problems on the basis of the fundamental principles involved.

Silver (1979) found that successful problem solvers were far more likely than unsuccessful ones to relate and categorize mathematics problems on the basis of their underlying similarities in mathematical structure. In judging problem similarity, unsuccessful problem solvers were more likely to rely on surface similarities in problem setting or context or on the question asked in the problem.

In the study of algebraic problem solving, several investigators have noted the usefulness of schema theory in providing phenomenological explanations. For example, Hinsley, Hayes, and Simon (1977) and Mayer (1982) have argued that routine problem-solving performance with stereotypical algebraic word problems can be explained in terms of students' schemas for the problem types. In particular, they concluded not only that their subjects had schemas for standard algebra problems but also that the schemas influenced the encoding and retrieval of information during problem solving.

Schemas are useful not only for retrieving clusters of related and useful information, but also for shaping the representation of problems. In research dealing with young children's ability to solve arithmetic and subtraction story problems, for example, the differences between highly skilled and less skilled performance by children has been modeled in terms of the development of more powerful problem schemas for representing the problems (e.g., Riley, Greeno, & Heller, 1981; Briars & Larkin, 1984).

From a cross-cultural perspective, it might be productive to examine the typical schemas that characterize the thinking of successful and unsuccessful students in different countries. Since stereotypes vary across cultures, one would expect that the schemas would also vary. Such a study could suggest important implications for the design of instruction in each country.

Meta-processes

We have already seen that the cognitive science perspective suggests that extensive domain-specific knowledge appears to be vital to success in problem solving. It is reasonable to ask what other kinds of knowledge are involved in skillful problem solving. Several cognitive researchers interested in mathematical problem solving (e.g., Garofalo & Lester, 1985; Schoenfeld, 1985; Silver, 1985) have argued for increased attention to metacognitive aspects of the problem-solving process. These processes - such as initial assessments of personal competence or problem difficulty or managerial decisions regarding allocation of cognitive resources - often appear to be the "driving forces" of a problem solution episode. According to Flavell (1979), metacognition refers to one's knowledge of one's own cognitive processes and products, and of the cognition of others. It also refers to the self-monitoring, regulation, and evaluation of cognitive activity.

Metacognition is not a new construct; provision for metacognitive functioning has been made in most information-processing models of cognition. Atkinson and Shiffrin (1968) used the term "control processes" to identify certain voluntary and strategic behaviors that help one remember like rehearsing a telephone number or tying a string around one's finger. Butterfield and Belmont (1975) posited the existence of an "executive function" that selects appropriate "control processes" on the basis of task and environmental constraints.

Although many models of human information processing include metacognitive processes, up until recently there had been relatively little empirical investigation of them. Schoenfeld (1985) has conducted the most extensive inquiry into the self-regulatory control processes of planning, monitoring, and evaluation. He has shown that many problem-solving failures can be directly linked to failures to monitor problem-solving behavior. He has also demonstrated that students can be taught to be more attentive to their problem-solving actions and monitor them more effectively.

Lester and Garofalo (1982, 1985) have studied the extent to which students are "aware" of problem-solving processes that they might use. Their research suggests that students are largely unaware of problem-solving processes and that specific attention to this issue may be an important component of problem-solving instruction.

Also of interest is the role that beliefs about mathematics or about problem solving play in skillful problem solving. Although Norman (1981) has identified "belief systems" as one of the important issues for cognitive science and although Carbonell (1981) has constructed a mechanism that incorporates knowledge contained in belief systems (in this case, political ideologies) into the process of formulating plans for action, researchers working on problem solving have tended to ignore the role of beliefs in skillful problem solving. It seems clear that no process model of problem solving in any domain can be complete without an adequate account of the role of metacognition and belief systems.

Schoenfeld (1985) and Silver (1982) have argued that a problem solver's mathematical beliefs may be the "driving forces" in many problem-solving episodes. For example, beliefs that "there is always a rule to follow in solving a mathematics problem" or that "mathematics problems can always be solved in five minutes or less" can be powerful mediators of problem-solving performance.

The cross-cultural study of beliefs and attitudes regarding mathematics would be an important component of a cross-cultural research agenda. The Second International Mathematics Study can provide some data, but further clinical study should be undertaken to examine in detail the beliefs of students in different countries about mathematics, the nature of learning mathematics, and the processes involved in doing mathematics. Such a study would provide important information about the final product of the mathematics education systems in each country; such information would complement the achievement data already available.

PROBLEM-SOLVING INSTRUCTION

As we noted at the outset, the teaching of problem solving is regarded by many as the most important and fundamental goal of school mathematics instruction. In this portion of the paper, we consider briefly a few of the findings directly related to the teaching of problem solving. Cognitive theory and research suggest that problem-solving expertise depends on extensive domain-specific knowledge; thus, we examine those instructional approaches that emphasize domain-specific knowledge. Nevertheless, other approaches to the enhancement of problem-solving ability have also been taken, and they are considered as well.

Domain-Specific Approaches

Marcucci (1980) examined the findings of 33 research studies conducted in elementary school classrooms since 1950. Using a statistical technique known as meta-analysis, which allows for the quantitative comparison of different studies on the same topic, Marcucci concluded that heuristic teaching methods (those emphasizing general heuristic problem-solving strategies and skills) were more effective than other instructional approaches.

Much of the recent work in this area has been greatly influenced by the writings of the eminent mathematician George Polya (1962, 1965, 1973). Polya proposed a four-phase model for problem solving - (1) understand the problem, (2) devise a plan, (3) carry out the plan, and (4) look back - and emphasized the importance of heuristic thinking and reasoning at each phase of his model. Although Polya's model is clearly deficient as a description of actual problem-solving behavior, it has been useful in suggesting ways to organize instruction to promote improved problem-solving performance.

Most of the research on the teaching of general problem-solving skills and strategies has been conducted with secondary school and college students. For example, Kantowski (1977) demonstrated that heuristic instruction could be very effective in enhancing the geometry problem-solving performance of secondary school students and Goldberg (1974/1975) found that college students could profit from heuristic instruction.

There have been a few studies that have shown the feasibility of heuristic instruction with elementary school children. For example, Lee (1978) was able to teach fourth graders to use general heuristic strategies, such as making tables or drawing diagrams. Putt (1979) found that fifth-grade students could benefit from heuristic instruction so that they were able to use many of the strategies, they developed an appropriate vocabulary for discussing their strategies, and they were able to suggest many questions appropriate for understanding a problem.

Metwali's (1979) work suggests the importance of teacher-student discussion as a mediating factor in learning from problem-solving instruction. In fact, Metwali found that the combination of giving students a few problems to solve and discussing the solutions in class was more beneficial than merely giving students many problems to solve. Other current research suggests that student-student discussion may be an important facilitator of growth in problem-solving skills and strategies. Noddings and her associates (Noddings, 1985; Noddings, Gilbert-Macmillan, & Leitz, 1983) have found that having children work in cooperative small groups to solve mathematics problems can result in significant growth in problem-solving competence.

As part of a cross-cultural research agenda, it would be interesting to examine the current practices of teaching problem solving in the United States and in Japan. Polya's writings have received attention in mathematics education throughout the world, and they have substantially influenced the teaching of problem solving in the U.S. It would be interesting to compare the approaches taken to implement the "heuristic" approach in different countries.

Domain-Independent Approaches

There are a number of programs that have been designed to teach domain-independent thinking skills that can, when mastered, be applied to problems in any given domain. For example, two programs that comprise part of the school curriculum in many countries are de Bono's (1976, 1977) CoRT Thinking Program and Feuerstein's (1980) Instructional Enrichment (IE) Program.

One of the earliest attempts to teach children general problem-solving strategies and skills was the Productive Thinking Program (PTP) designed by Covington and Crutchfield (1965). Like CoRT and IE, PTP was aimed at developing "master thinking strategies" that could be applied to any content area. In research studies involving PTP, it was found to be an effective program for (a) promoting divergent thinking, originality, and perceptions of the value of problem solving for fifth- and sixth-grade students (Covington & Crutchfield, 1965), (b) improving the divergent thinking and problem-solving performance of fifth-grade students (Olton, Wardrop, Covington, Goodwin, Crutchfield, Klausmeier, & Ronda, 1967), and (c) developing verbal creativity and problem-solving skills in children in Grades 4-7 (Treffinger, 1969). However, attempts to show "transfer" of training from the predominantly verbal PTP to the domain of mathematics were generally unsuccessful (Jerman, 1971; Treffinger, 1969).

In general, the research related to general thinking-skills programs appears to suggest that the programs can be successful in improving student performance only on tasks that require limited domain-specific knowledge (e.g., Whimbey & Lochhead, 1984). Since mathematical tasks typically require extensive domain-specific knowledge, improvements induced by general thinking-skills programs appear to have little specific effect on mathematical problem-solving performance.

CURRENT STATE: EMERGING THEMES

This paper concludes with the brief mention of three themes - affect, assessment, and technology - that I believe will emerge as important future trends in research on mathematical problem solving. Although they are not dealt with in much detail here, I expect them to become dominant research emphases during the next five to ten years. Since they are all themes that bear heavily on the school context in which mathematics education occurs, some aspects of these themes are discussed more fully in other papers prepared for this conference.

AFFECTIVE FACTORS

There is general agreement that affective factors, such as motivation, interest, self-confidence, anxiety, and perseverance, play an important role in problem solving. Yet we have little conclusive information about their influence on mathematics performance in general or on problem solving in particular. In a recent review of research on mathematical problem solving, Lester (1980) indicated that, "after a careful review of the literature on problem solving and a year of observing over 700 intermediate-grade children solving problems, the staff of the Mathematical Problem Solving Project (MPSP) decided that willingness, perseverance and self-confidence were three of the most important influences on problem-solving performance...However, the MPSP was unable to develop an attitude instrument to measure adequately the extent to which these three factors changed over time, even though the staff and classroom teachers were confident that very definite changes had occurred" (p. 299).

As Lester noted, the lack of information about affective factors is due, in part, to the difficulty of designing instruments that can reliably measure these factors. Nevertheless, it may also be that our "cognitive blinders" have not allowed us to examine these affective factors when they have appeared in our research.

Despite the shortcomings of previous research in this area, there now appears to be considerable enthusiasm in the research community for studying the affective influences on problem-solving performance. For example, McLeod (1985) has suggested the development of a theoretical framework for the study of individual problem-solving episodes that would incorporate both cognitive and affective factors, and he is currently engaged in that project.

ASSESSMENT

Given the widespread interest in improving the problem-solving performance of precollege students, it is natural to ask how the improvement will be demonstrated. Standardized tests are widely used in American education to provide measures of educational improvement and change. Unfortunately, current standardized tests favor students who have a large store of facts, definitions, and routine skills, and the tests do little to assess students' abilities to use that knowledge to solve problems (California Mathematics Council, 1985; Resnick, 1986).

Many programs that seek to improve the problem-solving ability of students are hampered by the lack of adequate assessment techniques to measure the cognitive, metacognitive, and affective changes that result from the program. How can a standardized test measure a student's improved thinking skills that may result in improved performance two years hence? How can the tests measure the increased willingness of students to engage mathematics problems, the increased confidence in their ability to solve them, or the feelings of increased self-worth that result from the experiences provided in the problem-solving program?

Some states have included special items dealing with higher-order thinking skills in their state testing programs. For example, California has developed a set of innovative problem-solving items as part of the California Assessment Program (1985; Pandey, 1986). These items test the non-answer-giving aspects of problem solving and are written for each of four components of problem solving: problem formulation, problem analysis, problem-solving strategies, and problem interpretation. Although these items, and similar ones used by some other states, are an improvement over the problem-solving portions of commercially published standardized tests, they are quite limited in scope and do not provide sufficient opportunity for students to engage either in extended investigation or in the solution of open-ended problems.

It is clear that considerable work will be required in order to produce assessment techniques that are truly sensitive to the cognitive, metacognitive, and affective changes that may be involved in improved problem-solving ability. The development of new testing approaches should proceed from a solid research base on the nature of skilled mathematical problem solving, it should build from and incorporate some of the techniques used by researchers (e.g., situational problem-solving tasks, interview settings), and it should include a substantial research component concerned with analyzing performance on the new items.

TECHNOLOGY

Despite the fact that much has been written about the impact of technology on the mathematics curriculum and the mathematics classroom, there has been little research done on the specific ways in which the technology might transform mathematical thinking and problem solving. But there are currently a few projects that are beginning to investigate the cognitive (and perhaps metacognitive and affective) impact of computer technology on precollege and college students.

Computers can have a substantial impact on the study of mathematical problem solving because they can provide new problem-solving environments (such as microworlds), they can enhance a person's problem-solving capabilities through the availability of software "tools," and they can augment a learner's cognitive capacity by simultaneously displaying graphical, numerical, and other symbolic representations. These features should all be important topics for future research on problem solving.

There are other technologies, such as video and video-disk, which may also significantly alter the fundamental character of mathematics education. The possibility of presenting large amounts of information to students in the form of dynamic visual images could substantially alter the nature of mathematics learning as we now know it as a product of static text-based presentations.

The impact of technology on school environments in the next two decades is not clear, but it may be substantial. As we prepare for mathematics education in the twenty-first century, it is not at all clear that most of what we know about human problem-solving performance on paper-and-pencil tasks will hold true in technologically enhanced environments. A substantial research agenda needs to be set.

CODA

As we have seen, recent trends in research on mathematical problem solving in the United States have involved the examination of individual student knowledge and performance in detail, to attend to individual differences within groups of mathematical problem solvers, and to use qualitative and clinical methodologies. These tendencies have not been as evident in Japanese research on mathematical problem solving, which has tended to focus on group rather than individual performance. Clearly, much work lies ahead in establishing a common set of questions and procedures for cross-cultural studies of mathematical problem solving, but the rewards will surely make the effort worthwhile.

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Discussion of Professor Silver's Paper

Miwa: Thank you, Professor Silver. In a very limited time, you have covered a great deal of content. Now we will have discussion.

Hashimoto: I have three questions. I understand there are many meanings of problem solving. My first question is what do you think of problem solving, problem making or problem posing?

Silver: Regarding the question of problem posing, problem making, and problem formulation, that general topic is one that is largely unresearched in the United States. It is a topic which has been discussed at many meetings and discussed in many ways in articles, but the questions of where do problems come from, when is a problem your problem, when is it my problem, and when is it our problem are central questions to understanding performance in this area that have have not been carefully examined. There is a recent paper that Jeremy Kilpatrick wrote on this topic of problem formulation. In it he reviews only a very little research that he could identify that deals directly with that area. There is in the history of mathematics certainly some information about where problems come from, the importance of metaphor, imagery, and so on, but there is very little research on two fundamental issues: one is how students generate problems of their own and secondly, and maybe more importantly, how we transfer ownership of the problem from the teacher and the curriculum developer to the student so that it becomes the student's problem. That's only question one. Do you have question two?

Clarkson: Although there is not much research available on problem posing, one of the things that we will see in the new textbooks that are just being published this year in the United States is that most of them contain several different sections that relate to this question. In some sections there is simply a story or a situation that is set up and students are then encouraged to make up their own problems or to determine whose problem is being presented. Many kinds of data are given and the students are encouraged to ask a question that they can solve with this data. Different real-life problem situations are also given and the students solve them as a class or small group. These are being included in almost all of the new textbook series that are being published.

Hashimoto: My second question -- Dr. Silver has said that there has been more than 10 years of collaboration between psychologists and mathematics educators. My question has to do with the role of mathematics education researchers. What do you think of their role?

Silver: The process has been, I think, a process of mutual education which has occurred through various conferences and seminars that have been held. In general, we are now at the point where there are some psychologists and some mathematics educators who can talk to each other and have a common language through which to communicate fairly clearly. I still think that, in general though, the mathematics education community and the psychology community in the United States do not view problems in the same way. The mathematics education community tends to view, and give obviously much more importance to the mathematics that is being dealt with in the particular setting that is being discussed, whereas the psychologists tend to use mathematics as only an area in which to study human thinking. Yet there has developed a group in which there seems to be more interchange - the psychologists have begun to pay more attention to the mathematics and the mathematics educators have begun to pay more attention to the psychology. The influence is more on each individual's work than on large scale cooperative projects. It's more that individuals have been influenced by individuals outside their field than that there have been teams of math education and psychology people working together.

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Hashimoto: My last question is about problem solving expertise. Professor Silver pointed out five areas. I think in Japan there are such situations indicating that the collaboration between math educators and psychologists is increasing. My third question is what do you think of these five aspects on page 42?

Silver: There is a very large literature available that deals with problem solving and these particular subtopics I used were just convenient categories around which I could organize a large amount of information. I think that they capture, both in the psychological literature and in the mathematics education literature, much of what the emphasis has been. The notion of representation is central to the work in both fields. And whereas pattern recognition and memory schemas, for example, have been more prevalent in the psychological literature, some researchers in mathematics education have begun to take those ideas and point out that many of the phenomena in which we are interested in mathematics can also be described in terms of those theoretical constructs. So these five were merely convenient categories for organization of the discussion.

Nohda: Dr. Silver first takes pattern recognition as one of the categories of problem solving. As far as pattern recognition goes, I understand somewhat what are the psychlogical studies or psychologists' standpoints in this area, but I am not sure what is the role of mathematics education researchers. Where is this role? The second part of the question is in regard to understanding in the business of double meaning, the output understanding and this inner understanding. How much research or how is research being done in the United States in this area today?

Silver: I would like to answer the questions in the reverse order. As far as the understanding is concerned, at least the contrast could be made between internal understandings and external understandings and there are at least those kinds of differences. There are actually many levels of understanding,

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different kinds of understanding which are referred to in our literature with the same term, understanding or meaning, and actually I think there is a fair amount of confusion that results from that or at least from imprecise use of language. There are aspects of understanding that have been examined, including the so-called "understanding the problem." The statement of the problem, the conditions, the constraints that problem imposes and the nature of the task that is at hand for the solver is one kind of understanding. Another kind of understanding is concerned with the relationship between the mathematical operations that one performs, the relationship between that mathematics and the problem task itself, and the answer and how that is to be interpreted back in the problem situation. And then there are understandings that are associated with the actual performance of the task - When I perform a certain algorithm, do I understand why I perform it in the particular way? Do I understand why I make marks in a certain place on the paper and what do they mean? What understanding do these carry? So there are all of these kinds of understanding which are involved in the word understanding and all of them have been studied to a certain extent. However, there is no source that I know of which clearly delineates among those different kinds of understanding or even what the relationships might be among them. Concerning the pattern recognition question, there are many aspects of expert behavior which appear, in mathematics, to be the result of the solver recognizing certain patterns, patterns in the symbols, or structural patterns in the problem task. These then generate certain responses from the problem solver and then this pattern recognition becomes a phenomenon of interest in terms of studying the way in which the knowledge that the expert problem solver has is organized internally so that it can be retrieved in appropriate ways at those moments when these patterns occur. That largely is a matter of extensive experience in solving problems in certain task areas of mathematics.

Nohda: I did not hear in your presentation or see in the paper footnotes of continuing research. If you could provide me with bibliographical sources after the presentation, I would appreciate it very much.

Senuma: I have two questions: Are the people working in problem solving today the same researchers that did much work with the new mathematics or are they entirely different people?

Silver: Even if they are the same people, they are surely different people now. By and large the people who created the new mathematics material and were responsible for the generation of that curriculum are not now involved in current research in the area of problem solving. There are some people, however, who were involved in studying the effects of the new mathematics like Dr. Wilson, Dr. Becker, and others from Stanford University who are still actively involved in studying problem solving and whose doctoral students are still actively involved in doing current research in the area of problem solving. In that sense the strand still continues.

Senuma: The second question is about influence of people like Brownell, John Dewey, and Polya in the present research in problem solving. Do they have any influence or impact on the work?

Silver: Yes, definitely, a very strong influence. Polya's influence is probably the most direct and apparent. There must be a hundred or more studies that have been done that directly relate to Polya's organization for instruction in mathematical problem solving. Those studies, in terms of actually being based on Polya's writings, are becoming a little less frequent now than they were 10 years ago. But the influence is clearly there. In some sense the research is now moving to the next stage beyond the simple list of heuristic processes which would be useful in studying more clearly the particular characteristics of individuals and tasks and how these processes interact with individuals. So Polya's influence is very profound. Regarding Brownell, Dewey and some others, the influence is a little more subtle, but it is also there.

Ishida: In listening to your presentation, I detect two currents of thought, one is Polya's way which relates directly with pure mathematics research and the other is Dewey's which relates more to real world research. Of the two, which do you feel is more applicable?

Silver: Are those my only two choices?

Ishida: You could include more.

Silver: I think that we will make progress in understanding mathematical problem solving to the extent that we attack the questions from as many different perspectives as we possibly can. I think there is plenty of room for work to proceed along many fronts. It seems that there is a considerable amount of interest in real world tasks, and I am personally interested in how people make contact between mathematics in school settings and mathematics in out-of-school settings. But I and most of my colleagues in mathemtics education are also very interested in how people perform within the discipline of mathematics itself and both are quite important and there is room to proceed in both.

Miwa: This is now the end of discussion. Thank you.

SESSION 2

Professor Wilson's Paper

Shimada: I will now introduce Professor Wilson. I know him very well through the Second International Mathematics Study of IEA. Once its meeting was held in Tokyo, and on that occasion I talked with him about various aspects of mathematics education. As you all know, he is an active researcher in this field, so let's listen to his presentation.

Wilson: Let me start off with a slight correction because I view myself primarily as a practitioner of mathematics education and this paper is written from the point of view of a practitioner. I think I can converse about research and I have ideas of what research might be needed, but the topic as I saw it was one of dealing with what my view of classroom practices are in the United States at this time with problem solving and it is written from my perspective as a practitioner in mathematics education.

CLASSROOM PRACTICE OF PROBLEM SOLVING IN U.S. CLASSROOMS

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Introduction

My topic, classroom practice of problem solving in mathematics in the United States, certainly provides a broad scope of issues to address in this paper and a freedom to select from alternatives. It would be unrealistic and presumptuous to try to be comprehensive in this task. Rather, I will present a general background and give my perspective on the diverse situations regarding problem solving in school mathematics in the U.S..

I have had the opportunity this year to be involved in several assignments dealing with school mathematic programs and inservice teacher education. These have included 1) a comprehensive study of the mathematics program of a large Maryland school district having over 25,000 students, 2) an accreditation team service for a rural Georgia school district with less than 1,000 students, 3) leading a year-long project of 19 inservice teachers in the study of problem solving in mathematics and the implementation of problem solving into their on-going teaching assignments, 4) participation in the planning, initiation, and instruction for a three-year project to prepare leader teachers in Tennessee to implement problem solving in their schools and to provide assistance to peer teachers, 5) directing a national task force on testing and evaluation in mathematics, 6) participated in the revision of mathematics teacher certification criteria for Georgia, 7) offered my course, EMT 525/725 Problem Solving in Mathematics, four times, and 8) supervised nine student teachers or interns during the year. Each of these, and other activities, will make its way into the following discussion. They provide my contacts, along with general professional involvement, with classroom practices in mathematics problem solving.

Agenda for Action, etc.

The National Council of Teachers of Mathematics, with its approximately 50,000 members, is a major voice in mathematics education. A four-year project of the NCTM Board of Directors to prepare a statement on school mathematics led to the publication of An Agenda for Action: Recommendations for School Mathematics of the 1980s in 1980. This document had come from discussions and work from throughout the NCTM membership. It was based on the work of several committees and surveys in the U.S. and Canada on priorities in school mathematics.

The Agenda was presented at the NCTM Annual Meeting at Seattle in 1980. It was intended as a discussion document. It raised issues for attention by the whole U.S. mathematics education community. The Board of Directors called for "all interested persons and groups to join us in a massive cooperative effort toward better mathematics education for all youth" (NCTM, 1980, p. ii). I was a member of the Board and the Board Committee that directed preparation of the Agenda. There was clearly a broad base of support for the substance of the recommendations to follow, but plenty of debate over strategies on statement. The title came late in our discussions because we came to agree that the document could not be prescriptive. Rather, it was to set an agenda for discussion, development, and growth for the coming decade.

It is perhaps instructive to ask how the agenda is going now that we are two-thirds of the way through the decade. This could be interpreted as the task for my paper and I will give some opinion on that point. First, however, the major recommendations of the Agenda should be noted.

There were eight major recommendations in An Agenda for Action. These were as follows:

1. Problem solving must be the focus of school mathematics in the 1980s.

2. The concept of basic skills in mathematics must encompass more than computational facility.

3. Mathematics programs must take full advantage of the power of calculators and computers at all grade levels.

4. Stringent standards of both effectiveness and efficiency must be applied to the teaching of mathematics.

5. The success of mathematics programs and student learning must be evaluated by a wider range of measures than conventional testing.

6. More mathematics must be required for all students and a flexible curriculum with a greater range of options should be designed to accommodate the diverse needs of the student population.

7. Mathematics teachers must demand of themselves and their colleagues a high level of professionalism.

8. Public support for mathematics instruction must be raised to a level commensurate with the importance of mathematical understanding to individuals and society. (NCTM, 1980, p. 1):

Each of these major recommendations was accompanied by a brief discussion and a set of recommended actions. For example, the recommended actions for the first Agenda item were the following:

1.1 The mathematics curriculum should be organized around problem solving.

1.2 The definition and language of problem solving in mathematics should be developed and expanded to include a broad range of strategies, processes, and modes of presentation that encompass the full potential of mathematical applications.

1.3 Mathematics teachers should create classroom environments in which problem solving can flourish.

1.4 Appropriate curricular materials to teach problem solving should be developed for all grade levels.

1.5 Mathematics programs of the 1980s should involve students in problem solving by presenting applications at all grade levels.

1.6 Researchers and funding agencies should give priority in the 1980s to investigations into the nature of problem solving and to effective ways to develop problem solvers (NCTM, 1980, pp. 2-5).

I will forgo listing the recommended actions of all of the Agenda items. Later, I will select a few for discussion. These clearly vary in level of specificity, degree of clarity, and likelihood of being implemented. Indeed, whether any particular recommendation was "clear" or "implemented" was not an issue. The issue is whether the recommendation has led to the involvement of "persons and groups...in a massive cooperative effort toward better mathematics education" (NCTM, 1980, p. 11).

There was not much that was original or creative in An Agenda for Action. Indeed it reflected informed judgement from throughout the profession. It reflected compromise. It ignored some dissenting opinions. Many important developments in the field preceded (and followed) it.

The National Council of Supervisors of Mathematics had declared problem solving to be a basic skill in 1977 and stated that

> Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving but students also should be faced with non-textbook problems. Problem solving strategies involve posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error. In solving problems, students need to be able to apply the rules of logic necessary to arrive at valid conclusions. They must be able to determine which facts are relevant. They should be unfearful of arriving at tentative conclusions and they must be willing to subject these conclusions to scrutiny. (NCSM, 1977, p.2)

This NCSM position statement is noteworthy because it was produced at the height of the "Back to Basics" movement. It came about as a specific response to the report of the National Institute of Education Conference on Basic Mathematical Skills and Learning (The Euclid Conference Report) held in 1975 at Euclid, Ohio. I am struck by an atmosphere of two cultures (with apologies to C.P. Snow) where even though everyone was talking about "basics" one group meant arithmetic skills and the other meant problem solving, applying mathematics to everyday situations, alertness to reasonableness of results, estimation and approximation, appropriate computational skills, geometry, measurement, reading, interpreting, and constructing graphs, tables, and charts, using mathematics to predict, and computer literacy. There were a few informed people outside of mathematics education who understood the NCSM statement on basic skills but for the most part it was a document for those of us on the inside.

Hawaii and Georgia issued new state curriculum guides in mathematics in 1980 that had considerable parallel with the Agenda. Mathematics staffs in these states were at work long before 1980 to prepare their guides. Other instances of mathematics education activity consistent with the Agenda (or vice versa) could be easily cited.

In England, The Committee of Inquiry Into the Teaching of Mathematics in Schools, with W. H. Cockcroft as chairman, began its work in 1978 "to consider the teaching of mathematics in primary and secondary schools in England and Wales, with particular regard to the mathematics required in further and higher education, employment, and adult life generally, and to make recommendations" (1981, p. ix). With due respect to style and substance of the report titled Mathematics Counts, there is considerable parallel among the problems, the recommendations, and the progress so far in our two countries.

Perhaps the 1980s will go down in the history of U.S. education as the most studied, most analyzed, and most mis-directed by national reports and national conferences. Lee Shulman noted in 1983 that there were 32 reports released or under preparation that had been called to his attention. The most celebrated (infamous?) of these was A Nation at Risk.

A Nation at Risk

The National Commission on Excellence in Education released its report in 1983. The specifics of this report do not need to be brought up here. My main concern is to note the extensive media attention given to this report and reactions to it. Its strong negative rhetoric struck a responsive chord among the public. The report had many positive suggestions for improving education but these lost visibility among the critical material.

The positive aftermath of A Nation at Risk rested in the broad, new awareness among the public of needed improvements in education. Mathematics and science achievement were particularly noted. There was renewed acceptance of the importance of mathematics and science by the public and some orientation to provide more support. The Commission's recommendation for three years of high school mathematics for every student was certainly consistent with NCTM's Agenda recommendations. Additional studies and reports were issued by the Department of Education's Office of Educational Research and Improvement (Romberg, 1984) and the National Science Board Commission on Precollege Education in Mathematics, Science, and Technology (NSB, 1982), the College Entrance Examination Board (1983), and the Conference Board of the Mathematical Science (1982, 1984). These reports served various purposes and had different emphases but all served to raise alarm about the need for improvement in school mathematics and all reinforced problem solving and uses of mathematics as important elements in improved mathematics programs.

Progress on the Agenda

The Agenda has had a big effect on the profession. Problem solving has become a major topic of discussion at professional meetings, it is part of the sales pitch for every textbook publisher, and it is included in almost everyone's goals for mathematics teaching at every level. Clearly, problem solving is the buzz word of school mathematics in the 1980s.

One of the positive features of this situation is that teachers throughout the profession have become involved in thinking about problem solving. In this sense problem solving is a grassroots phenomenon of school mathematics. The seeds are being planted. It remains to be seen whether there is enough nourishment present for flowers to grow. The school mathematics curriculum is very barren. Many good intentions of teachers toward problem solving cannot be nourished into practice when there are strong pressures brought on by testing, coverage of material, and limited vision of what mathematics can be.

Although there has been widespread attention to some of the Agenda items, some have been less visible. The improvement of mathematics education requires attention to all of them. In particular, lack of attention to finding a wider range of measures than conventional testing to evaluate mathematics programs and student learning literally makes problem solving instruction unlikely.

I feel that insufficient attention was given to three major difficulties in mathematics teaching when the Agenda was being written and subsequently. These three difficulties were 1) the effect of testing and assessment on mathematics programs, 2) the preparation of teachers, either preservice or inservice, to teach problem solving, and 3) the professional atmosphere of schools.

Since 1980 the demands for evaluation and accountability have increased. Testing is driving our curriculum. By the nature of its items, the curriculum driven by testing leads to vocabulary, memorized material, and algorithmic procedures. It tends to be a curriculum with minimal problem solving. More time on task for practice of such material will lead to improved performance but is this what we want?

The preparation of teachers to teach problem solving is a challenge involving their beliefs about the nature of mathematics, their beliefs about the nature of school mathematics, their knowledge of mathematics content and processes, and their ability to solve problems. I believe we must expand the idealized role of the teacher beyond that of mathematician and pedagog to include curriculum developer and researcher.

The professional atmosphere of the schools - the conditions of teaching - make mathematics teaching a less attractive career today than what it was 15 years ago. The reasons for this need to be understood and reversed.

A "Case Study"

The Howard County schools in Maryland are situated between Baltimore and Washington, D.C. There are 25,000 students in a large formerly rural county that is rapidly becoming suburbanized. The mathematics program was studied this past year by the staff, outside consultants, and an independent committee of 30 citizens. Completely separate and independent reports were prepared by the consultants and the citizen's committee. My report dealt with the program from grades 7 to 12.

Many on-site visits were made to the schools by the consultants and committee members. Each consultant visited classrooms in each school and each committee member observed at least 5 classes. Problem solving was seldom seen. I observed a lively discussion among eighth graders on the construction of a geometric proof. Most classes, however, were teacher explanation of the text followed by assignments from the text. Open exploration was observed in a few classes of the grade 4-6 talented and gifted program and in early grades classes with manipulatives.

Testing overwhelms the instruction in this school district. There are tests mandated by the State Board of Education, Maryland Functional Literacy Tests mandated by the legislature, and tests mandated by the district. The citizen's committee reported that in the elementary grades, all or part of 56 different days were taken for testing (in addition to the teacher's tests). Thus, one issue is the amount of instructional time taken from the curriculum. The lack of instructional time is one of the excuses given by teachers for not doing problem solving activities.

Testing affects the curriculum in a different way as teams of teachers work to update and revise county curriculum guides. One issue always present is the need to organize a course so that students can do well on the tests. Several teachers stated this concern to me.

The citizen's committee report made strong recommendations in support of mathematics program revisions emphasizing problem solving. One layman wrote a five-page paper on the nature of problem solving, citing Polya. They also called for extensive inservice programs to help teachers develop the background to teach problem solving.

In this school system, therefore, I see the effects and failures of the Agenda firmly represented. There is an awareness among staff and teachers of the importance of problem solving in mathematics teaching but little implementation. The barriers to implementation include limited instructional time, need to cover material that may be tested, and limited knowledge of how to teach problem solving.

Another "Case Study"

Last March I participated in the site visit of the Southern Association of Colleges and Schools accreditation visit to Washington Wilkes Comprehensive High School. WWCHS is the only high school in Wilkes County, Georgia, and it enrolls around 500 students in grades 6-12. Here, under 30 percent of the graduates attend college whereas 85 percent of the students in Howard County attend college. About 50 percent of WWCHS students are black. It is not a wealthy district.

There were six mathematics teachers and I observed classes for each of them, talked with each of them, and developed some sense of the mathematics program. The teacher of a consumer mathematics class used a chalkboard simulation of a savings register to develop ideas of compound interest. The algebra II teacher used a number theory problem as an opener for his class. One sixth grade teacher showed me an essay written by a student describing selling stock in a lemonade stand to raise capital. The student wrote the paper independently after the class had studied a unit on stocks and a broker had visited the class.

In general, there were many recommendations for improvement of the mathematics program at WWCHS. The resources of the school are extremely limited. Five of the teachers are new to the system and the program needs to be updated. All six of the teachers are reasonably well prepared (e.g. over 60 quarter hours in mathematics). Two of them have been former students in my problem solving course.

Preparing Teachers for Mathematics Problem Solving

I believe there are several ingredients essential to a course or program of study that prepares a teacher of mathematics problem solving.

1. One must become a mathematics problem solver. Therefore my course begins with taking a large set of problems and selecting problems to solve.

2. Reflection about problems, solutions, extensions, attempts, etc. is an essential.

3. Modelling of problem solving processes is important. I work problems. I discuss strategies. If I go ahead and solve a problem I try to provide an extension.

4. It is up to me to make sure every student is occasionally successful in solving a problem.

5. Some problem solving should be in groups of 3 or 4.

6. Certain content can be developed through problems. Examples include Euler's formula, Pascal triangle from a street grid, triangular numbers, Heron's formula, the arithmetic mean - geometric mean inequality, harmonic mean, etc.

7. I like to use a few problem contexts with many extensions. For example, maximizing the area of a rectangle with fixed perimeter can be extended by considering regions other than rectangular shapes.

8. Throughout the course I am constantly looking for leverage to work on student's beliefs.

9. I regularly ask students to think about or write about how they would modify a problem to make it useable with their students.

A Problem Solving Workshop

During the winter of 1985 I proposed a workshop for experienced inservice teachers who wished to develop their background in mathematics problem solving, instructional computing, and (perhaps) the teaching of mathematics problem solving. I specified that the participants would be fully certified mathematics teachers currently employed in Georgia. This meant each participant had at least 60 quarter hours of mathematics and teaching experience. We selected 19 participants. There were several key ideas to the proposal.

1. A two-week August workshop would be given over to an intensive course in mathematical problem solving - with instructional computing whenever it was natural for a problem.

2. Each participant would develop an applied project to incorporate problem solving into their ongoing teaching assignment for the year. The project was to begin in August and be completed by May 15.

3. Five class meetings would be scheduled for Friday evening and Saturday morning (November, January, February, March, and May). The meetings were to assist with progress on the applied project, continue work on problem solving, and explore new topics.

Combining the problem solving course with the applied project allowed me to get a sense of the importance of "ownership." The projects were designed by each student to fit their teaching assignment. Not only were the projects their own creation, but there was the added benefit of using the material in their own class.

All 19 students finished the workshop. Their teaching assignment ranged from sixth grade mathematics to calculus. Their projects covered a similar span. Many of the projects were resource books to supplement the course they were teaching. All of them were able to document a successful implementation of problem solving. Further, most would not have gone nearly as far to implement problem solving without the applied project and its tryout.

MATHCAPS

This project at East Tennessee State University is directed by Dr. Katie Blackburn and although it is considerably more extensive than the workshop, it uses similar strategies of intense problem solving activity and development of ownership. Here the participants are writing materials to use in providing problem solving background for peer teachers in their schools.

Task Force on Testing

Growing concern for the effect of testing and assessment on mathematics programs has led NCTM to organize a Task Force on Testing. The following needs have been identified for the task force to address:

1. Identify standards for mathematics tests that include:
    - testing a comprehensive curriculum
    - emphasizing concept understanding
    - incorporating problem solving
    - assuming the use of a calculator
2. Analyze current tests commonly in use identifying matches and deficiencies to the test standards.
3. Develop a prototype for mathematics tests that includes sample items for various levels of students in a variety of mathematics content areas.
4. Discuss a variety of assessment practices that should be used by teachers in addition to formal written tests.

An open forum was held at the NCTM Annual Meeting in Washington, D.C. in April, 1986. Some discussion items from the session are given below.

Testing, Evaluation, and Assessment

Clearly stated goals of a mathematics program (curriculum) should dictate the content of tests used to evaluate program effectiveness; tests should not dictate the curriculum.

Tests to assess student learning must emphasize those priorities for mathematics learning that are identified in the stated curriculum goals.

Calculator use must be permitted on all competency/functional literacy tests for mathematics.

Testing for the purpose of instructional grouping must be frequent and accompanied by enhancement strategies.

Standardized assessments must reflect the changing goals of the mathematics curriculum with increased emphasis on conceptual development, creative problem solving, and higher order thinking skills and with decreased emphasis on algorithmic fluency.

Evaluation of student learning should permit alternative approaches that maximize identification of students' strengths rather than the assessement of students' weaknesses.

* The evaluation of mathematics learning should include a full range of the program's goals, including skills, problem solving, and problem solving processes.

* Teachers should become knowledgeable about, and proficient in, the use of a wide variety of evaluative techniques.

* The evaluation of mathematics programs should be based on the program's goals, using evaluation strategies consistent with these goals.

* Parents should be regularly and adequately informed and involved in the evaluation process.

* Evaluation strategies that include both test and non-test techniques should be developed and disseminated to mathematics teachers both in their preparatory programs and in continued in-service.

* The informed judgement of teachers should be considered a vital part of the evaluation of any student.

* Tests should measure what students know rather than what they do not know. (This is a call for greater use of criterion referenced tests.)

Summary

The classroom practice of mathematics problem solving in the U.S. is considerably more widespread than before 1980. Many teachers incorporate problem solving activities in their lessons. Yet, these classroom practices are far short of the level asked for in *An Agenda for Action*.

There are many factors inhibiting the transition to curricula with a problem solving focus. I believe much needs to be done to improve the preparation of teachers but there are some models around that show promise and the general grassroots nature of teacher discussions of problem solving facilitates.

I am much more pessimistic about the lack of progress we have made in dealing with the conditions of teaching or the effect of testing.

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Discussion of Professor Wilson's Paper

Shimada: Thank you, Dr. Wilson. Are there any questions or comments?

Travers: I have a comment, or perhaps a question. Jim mentioned the issue of teacher preparation in his paper and as you will hear tomorrow morning, and it will be of no surprise to anyone, when American teachers are asked about their goals and priorities, one of their primary goals is that of problem solving. They all agree that it is very important. But when we look at the data in terms of what goes on in the classroom, we see a wide discrepancy between their goals and practice. With respect to teacher education, it seems to me that it's expecting a lot (and perhaps it's an impossible expectation) that teachers will be fully, actively engaged in problem solving if they, themselves, are not engaged in problem solving as part of their OWN mathematics education. My question or invitation for comment from the Japanese side is: Can you provide any evidence that when your teachers are engaged in teacher preparation in their study of mathematics, do they have the opportunity to participate in genuine problem solving activities?

Shimada: How about a reaction on the Japanese side?

Nohda: Before I reply to Professor Travers' question, I have a question. I have read that Professor Larry Sowder has said in one of his papers, that there was no success in the United States in the teaching of problem solving. I would like to know whether this statement is true?

Wilson: It's probably true that Sowder said that. I think there has been progress, but my statement would be that if the *An Agenda for Action* represented 100%, we are now 10% of the way along. Are there other people in the U.S. delegation that want to take issue with me? Professor Sowder was looking at a very restrictive range of things called problems, as we all do.

Sugiyama: As we discuss this topic here, we should separate elementary and secondary schools. So about 10 percent of the elementary school teachers in Japan are mathematical specialists.

Wilson: 10 percent are not?

Sugiyama: 90 percent are not specialists in mathematics in Japan.

Wilson: A mathematics specialist in the elmentary school in the U.S. would be vary rare.

Nohda: That is an important point. So you have to separate the two cases - problem solving in the elementary school as contrasted with problem solving in our secondary school - because in secondary school, teachers have a professional license. In my opinion, at the school level, almost all teachers do not have enough time for teaching problem solving. Since the senior secondary school entrance examination is difficult for students, and the standards of the mathematics program are actually higher, the teachers may not emphasize problem solving.

Wilson: Well, I don't believe we stress problem solving either. That's exactly what Dr. Travers is asking. We ask for a lot of mathematical preparation but there is such a rush; we've got to get through the calculus to differential equations, through linear algebra, higher algebra and geometry. We cover so much material and do not have an opportunity for genuinely embedding problem solving in that.

Ishida: Although they have the mathematical preparation, teachers in the secondary schools of Japan do not pursue it in their classrooms either.

Wilson: What about their training in the university, do they pursue genuine problem solving in the university or do they work lots of mathematics?

Shimada: In this case, the problem is with the meaning of "problem" as a word. For senior secondary school teachers, it means those problems on the entrance examination, examples of which are included in the Appendix to these proceedings. They are very original, not routine, problems. But the solution is unique though there may be several ways to arrive at solutions. So secondary school teachers are sincere and hard working on such kinds of problem solving. Suggestions by Polya may help them. In these cases, the kind of problems is very limited in that they are not related to the real world but rather are confined within a mathematical context.

A few specialist teachers in the elementary schools have an interest in real-world problems. They devise very interesting and valuable problems. But most other teachers have no interest because real problems are more difficult to prepare and teach than the usual problems set by textbook authors. The difficulty lies in finding problems whose situation are enough interesting for students' emotion, and whose mathematical difficulty is suitable for students' levels of ability.

Next I will comment on teacher training. There are two kinds of training, preservice training and inservice training.

I have one good example of inservice training involving problem solving activity. A group of teachers was organized in the Osaka Science Education Center, an inservice training institution as well as a research institution supported by the Osaka Prefectural Board of Education, to concentrate on a study to help slow learners in senior secondary mathematics. Their first theme was fractions. In that study, they found and solved many of their own problems, some being psychological and statistical, and others being concerned with mathematical foundations. Through these kinds of activities they experienced what is problem solving by use of mathematics. This is the only case I know where problem solving elements are included in inservice training.

For preservice training, I now have one course for problem solving in mathematics in my university. In that course, I found many of my students do very well in solving given problems, but if I asked them a rather broader problem, it may be very difficult for them.

Wilson: I have an outline of my problem solving course here, and have had similar experiences.

Shimada: Are there any more questions?

Inouye: So apparently we are not talking about the same thing, when the question is posed about problem solving. Japanese participants may be thinking of some other kind of problem solving. Maybe the problem for Japanese teachers is how to get their students into Tokyo University.

Kantowski: I would like to ask Professor Shimada about that inservice training program in Osaka. What level are the teachers of that program, are they secondary school teachers of mathematics?

Shimada: Senior high school teachers.

Kantowski: So they are having some problem solving experience?

Shimada: They found many students in senior secondary school cannot do computation on fractions, and it was a teacher's real problem to devise a measure to remedy these students' difficulties. In the process, they experienced solving many subproblems. I suppose they might have some kind of problem solving experiences, but not so profound as the one mentioned.

Becker: It seems as though the Japanese and Americans have the same difficulties in training teachers to emphasize problem solving. And it sounds as though students have the same difficulties in solving problems in the curriculum, but yet the Japanese students seem to do so much better in the international achievement data. I wonder whether some people can make some comments that help to explain that.

Sawada: Yes, I am sure that Japanese students seem to show better achievement than that of other countries. But the reason is not clear to me. Another reason for good achievement may be that many students go to the "juku," informal schools. Thus, they attend two kinds of schools, and this tendency has increased every year.

Wilson: Do you regard those items on the IEA as problem solving? Are they problems in the sense that we are talking here?

Sawada: I will discuss this tomorrow.

Nohda: In Japan, the juku makes very strong emphasis on acquisition of computation. But I think that the juku schools (tutoring schools outside of regular schooling) are actually detrimental for problem solving. They stress calculations and so-called basic computational skills and actually interfere with what the public schools are trying to accomplish in problem solving.

Sawada: I don't think so. Both regular teachers and juku teachers do not like problem solving in the elementary through secondary school levels, so they don't stress it in their teaching. So many teachers don't like problem solving.

Shimada: May I interrupt here? In this case, there is a difference in the meaning of "problem." Problems involved in the IEA testing were an exercise type of problem, for which Japanese students had worked hard in preparation for examinations or other reasons. But our issue is: if the student can do well on such exercise type problems, then can we expect the student to also do well on a wider problem? My answer is negative. They can easily follow a set pattern, but when a set pattern cannot be found, they will be able to do nothing. The fact that the test score was higher than other countries may mostly be attributed to social pressure on students' learning. But this is not a good objective of mathematics teaching.

Travers: I would put the question another way, however, from the U.S. side. If students cannot even do the problems that are on the IEA test, are they ever going to be able to learn problem solving?

Shimada: In page 37 of Professor Silver's paper, there was a problem that dealt with the number of buses for soldiers. If this problem is discussed in the classroom both in Japan and the U.S. and then the results are compared, I think the U.S. students will more likely show a realistic attitude to the interpretation of remainder than Japanese students.

The expected correct answer was (integral part of 1128/36) + 1, (+1 for remaining 12 soldiers). But if we put these 12 soldiers in some of the other buses, more than 31 buses would not be necessary, and this may be a more practical solution in a real situation than driving one more bus for 12 soldiers. I think there may be a difference in favor of the U.S. between students in the two countries when considering the problem in such a practical context.

Wilson: The military answer would be to throw the extra one away (laughter).

Hashimoto: Generally speaking, what Professor Shimada says is true. But after Dr. Wilson's talk, I will show a video in which those kinds of ideas will appear. It depends on the teaching.

Wilson: I think part of the answer, my answer to Ken's question, is that as long as our curriculum is so driven by tests we can never have time to talk about real understanding. We emphasize memorization and drill on algorithmic procedures so that the students never have a chance to absorb reasoning and there are limits to how much they will grow. I think the students in our country are much more likely to answer "I don't care" to test questions. It's not part, however, of what I understand of Japanese culture for kids to say "I don't care" and refuse to do something, but it is part of the U.S. culture.

Kantowski: First, I would like to make a comment and then ask a question. I have a similar reaction to Ken Travers' question. The assumption is that you can't do problem solving until you have a mastery of computation and I'm not sure that is true. In developing problem solving skills, we can develop understanding that can lead to some amount of knowledge of computation that will be optimal or minimal for questions that come up on the international exam. I would like to ask a question of the Japanese delegation. Jim said that one of our biggest problems is that we are test driven and that one of the biggest impediments to the teaching of problem solving is that the teachers are concerned about having their students pass various types of tests. Does the Japanese delegation feel the same kind of concern about an impediment to teaching problem solving? So far, we have been emphasizing the fact that teachers have not been prepared in problem solving, that they are not doing problem solving; but my question is, is testing a driving force that would be an impediment to problem solving in your country as well?

Sugiyama: Japanese do not consider problem solving as a luxury. In the elementary school, there are not too many mathematically prepared teachers. But we do spend some time in problem solving.

We must look upon this problem from both within and without or the two sides, and one aspect to be studied is certainly the problem of curriculum and the textbook.

At least at the elementary level, Japanese teachers are less concerned with various aspects of the entrance examination. Some good teachers try to stress the teaching of problem solving and do spend some time in problem solving. Some teachers intend to develop understanding of arithmetic through developing problem solving skills or through developing reasoning abilities. But others may give only drill for algorithmic procedures. It is almost the same case at the secondary level too.

Becker: Jack and Elizabeth Easley from the University of Illinois spent some time in Japan a few years ago, spending perhaps three months in one school in Tokyo. They report in some of their writings what Professor Sugiyama has said. Though they are careful not to over generalize, they wonder whether on a larger scale in Japan the elementary teachers place more emphasis on thinking exercises for children as opposed to emphasizing computational type activities and that the results of that emphasis show up fairly clearly in the late elementary and junior high school years.

Miwa: I would like to make two comments. The first is based on the difference that I see between the opinions of Professsor Travers and Professor Shimada. It is whether students who have good scores in the exercise type problems are good solvers for real world problems or not. This may be an open problem for further study.

Next, I heard in the lecture of Professor Silver this morning that there is a multitude of meanings of "problem solving" in the aims of mathematics education. In Japan, the aim of mathematical education in the elementary and secondary schools is to help students foster a mathematical way of thinking. Problem solving is included in that. So when teachers teach many subject matters of mathematics, they make an effort to attain this aim; i.e., to help students develop mathematical thinking and to encourage or inspire them to solve difficult problems. Of course, these problems are those included in textbooks or those devised by teachers. Professor Shimada referred to problems which should be found in the real world or formulated by students themselves from their environment. So the difference will depend on what teachers see as a problem.

Shimada: I would like to add one example. About ten years ago, the results of an educational survey were reported. It said that the percentage of students who could master mathematics taught up to that grade were 70 percent, 50 percent, and 30 percent at the end of elementary, junior secondary and senior secondary school, respectively. These figures 70, 50, and 30 percent were regarded as indicating failure in mathematics teaching and were referred to as such in many articles on education. At that time, no mathematics teacher questioned the validity of the results and most accepted them as such.

But these figures can be interpreted by assuming exponential change in the following way:

Let 100x % be the percentage of pupils who are fully mastering mathematics in the previous grade and also succeed in the said grade. Then we will have $x^6 = 0.7$, $x^9 = 0.5$, and $x^{12} = 0.3$, assuming x is constant in average. These equations mean $x = 0.9$ or a little more. This means teachers made 90 percent success in each grade. (Marvelous work!)

What I would like to point out is that no mathematics teacher thought of such an interpretation, though most of them are good in problem solving of the type on examinations. So they cannot apply their abilities to such an easy real situation. Real world problem solving is very different from that found in the examination papers or in the IEA study. There are some delicate differences between real problems and those in conceptual or mathematical problems made by authorities.

Excuse me for making this interruption. As Chairman I spoke too much. Now it is time to adjourn.

Professor Hashimoto's Paper

Travers: Our next presenter is Yoshihiko Hashimoto who is Associate Professor of Mathematics Education at Yokohama National University. At the same meeting to which Mr. Shimada referred earlier I had the pleasure of first meeting Mr. Hashimoto, when he worked at the National Institute For Educational Research (NIER). Subsequent to that he came to the University of Illinois as a Visiting Scholar for the period of one year. At that time he taught us a lot about open-ended problem solving. So it is my pleasure to be able to preside over this session and to present Professor Hashimoto to you. He will give a paper on the topic of the classroom practice of problem solving in Japanese elementary schools.

Hashimoto: Thank you very much for your kind introduction.

CLASSROOM PRACTICE OF PROBLEM SOLVING IN JAPANESE ELEMENTARY SCHOOLS

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1. Introduction

The purpose of this presentation is to illustrate our experimental approach in teaching mathematics using a video image on screen, which portrays a classroom practice carried out by one of the collaborating teachers in our project. The experimental approach may be referred to as the problem posing approach, whose aim is to elicit children's mathematical ideas by asking them to pose similar problems after solving a given original problem. In addition, its aim is to solve a problem by choosing one which they feel to be challenging or interesting from such problems as are posed by their peers. This approach has been promoted by Sawada and others since 1978 (Hashimoto & Sawada, 1984; Nagasaki & Hashimoto, 1985) as a natural development of the approach using open-ended problems advocated by Shimada in 1971 (Hashimoto, 1983; Shimada (ed.), 1977).

The research project was undertaken to study the potential of this approach and to clarify its effectiveness as well as its pitfalls. We achieved a positive outcome, and are trying to disseminate the approach, but at present it is not very popular in Japan. The children's activity of posing problems similar to a problem just solved in class can be regarded as an important facet of problem solving activity in a broader sense since, in order to be able to pose a new problem, children must reflect on the essential structure or underlying abstract pattern of the solution of the original problem and transfer it to a different situation.

2. On the Lesson to Be Shown

The video-taped lesson is continued from the preceding one which focused on solving the following problem commonly found in Japanese elementary school textbooks.

Problem I

Squares are made by using matches as shown in figure 1. When the number of squares is 5, how many matches are used?

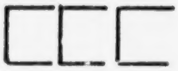


Fig. 1

The information on the class is as follows:

a. School: Matsubara Elementary School (public, located in a residential area of Tokyo)
b. Grade: Fifth Grade (10-11 years old)
c. Number of children: 37 (mixed, in gender and ability)
d. Teacher: TSUBOTA, Kozo (14 years experience)
e. Date: June 28, 1985

In this lesson, after reviewing the various ways of solving problem I presented by children in the preceding lesson, the children are asked to pose a new problem similar to problem I. Afterwards these problems are discussed, in turn, in the whole class as shown in table 1 below.

Table 1. Outline of the Lesson

Stage	Content	Time (min.)	Number in transcript
1	To present various ways of solving the problem and to review the last lesson.	6	1-15
2	To pose a problem based on the first problem.	15	18-23
3	To present to the whole class problems posed by individuals:		
	a. by changing the number or object.	3	25-31
	b. by changing the geometrical figure.	10	32-50
	c. by making converse problems.	11	51-71
4	To summarize.	2	72

Though the theme of this lesson is experimental and, therefore, not very common in Japan, the style of interaction between teacher and children is typical in Japanese elementary schools in that the teacher spends most of the time in discussion with the whole class and in observing or consulting with children who are working individually on the same assigned task. Individualized or group study on different tasks is not commonly used in Japan.

The whole lesson took about 50 minutes and was fully video-recorded. The present tape is edited from this fully recorded one to a 20-minute program for this Seminar. The first part (4 minutes) is composed of excerpts from instances of teacher-pupil interaction during the lesson in order to convey the overall atmosphere of the class. The second and third parts are recordings of classroom process emphasizing the following two aspects:

a. Recognizing and evaluating children's various ideas.

b. Discussions between teacher and child, and between children.

(Here the first part will be shown.)

(1) Recognizing and Evaluating Children's Various Ideas. (While Showing the Second Part)

In stage 1, table 1, eight ways of solving the problem (including the one wrong answer) were presented. The proposed solutions of the problem may be categorized as follows:

Table 2. General Ways of Solving the Problem

Fig. 4*	$4 \times \frac{n+1}{2} + 2 \times \frac{n-1}{2}$, n; odd	Fig. 8	$4 + 3 \times (n - 1)$
Fig. 6	$n \times 2 + (n + 1)$	Fig. 9	$3 \times n + 1$
Fig. 7	$n \times 3 + 1$	Fig. 10	$4 \times n - ((n + 1)-2)$

*Figures appear the first page of the Appendix and following.

Counting one by one in a systematic way is not included in Table 2, though basically it would be important for less able children. Besides these, there are many other ways, of which two may be noted here. One is that the frame of the rectangle is counted first, and then the vertical matches inside the rectangle are counted, or $(5 \times 2 + 2) + 4$, or $6 \times 2 + 4$. Another is to derive a functional relation by making a table representing the relation between the number of squares and that of matches.

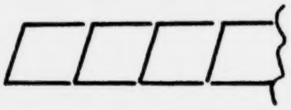
The teacher speaks with a tone that encourages responses and he recognizes the value of the children's ideas. For example, in the child's question "If the number of squares is larger, is it still 6?", the possibility of generalization about the ways of solving the problem is apparent. And the teacher encourages this line of thought by saying "- - -. If so, we may use this method in other cases." (No. 10, Appendix)

(2) Discussions between Teacher and Child, and between Children (While Showing the Third Part)

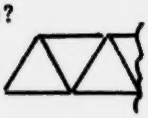
The converse problems shown below were used as the material of these discussions. The problem which follows is the converse to problem I above.

When the number of matches is given, how many squares can be made?

Problem A: Parallelograms are made by using pencils of the
(No. 51) same length as shown in the figure below.
When the number of pencils is 37, how many
parallelograms can be made and how many pencils
will remain?



Problem B: There are 27 sticks. Equilateral triangles are
(No. 62) made by using these sticks. How many equilateral
triangles can be made?



Next, the following six topics are discussed using the examples of the above two problems.

a. The Problem of the Term 'Parallelogram' (No. 52-54, Appendix)

Is the term 'parallelogram' in problem A appropriate? The use of geometrical terms for the figures in this problem is refined from parallelogram to square and to rhombus in turn.

b. The Problem of a Remainder ——— Case 1 (No. 55)

To the question, "What do we do if there is no remainder?", a child answers, "If there is no remainder, just say zero."

c. Correcting the Statement of the Problem (No. 57-62)

The problem was at first posed as below, and was revised to be like problem B.

> There are 27 sticks. Triangles, some triangles
> are made by using these sticks. How many triangles
> can be made?

d. Common Point of Two Problems (No. 63)

The number of pencils or sticks is given, and it is asked how many geometrical figures --- parallelograms in problem A, equilateral triangles in problem B --- can be made.

e. Reconsidering the Assumptions in the Problem (No. 64)

Though the term 'sticks' is used simply, if the sticks are too big, then the arrangement might not be possible. Defining the sticks can be seen also in No. 27-29.

f. The Problem of a Remainder ——— Case 2 (No. 66-71)

In problem B the question is "How many --- can be made?" What do we do if there is a remainder in problem B?

3. Teacher's Philosophy of Mathematics Teaching

Some teachers emphasize the value of the discussion between children and of children's ideas. It comes from their child-centered philosophy of mathematics teaching. This classroom teacher walks around the desks and cares about what and how children are thinking. This will help him determine his next action. Such behavior can be seen easily in the video recordings.

According to the Report about Tokyo-Hawaii junior high school math teachers (Whitman et al., 1986), it seems that Tokyo teachers walked around the desks more than Hawaii teachers.

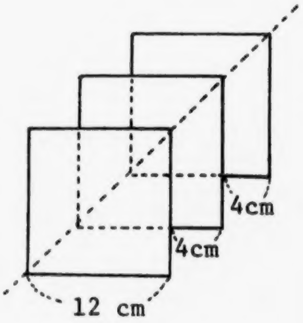
> Tokyo teachers more than Hawaii teachers monitored students' progress in learning and completing tasks by scanning the room to see if everyone was working, by monitoring students' responses, and by roaming the room checking students' work. (p. 24)

In this lesson, children pose the problems, but this is also due to the teacher's philosophy of mathematics teaching. As shown in "Who proposed it?" in Appendix A of Shimada's paper, posing a problem may be done by the children themselves or by an adult such as a teacher, the author of a textbook or an examiner. There are few cases in which children pose the problems.

In the following two problems, the first problem (problem II) is a given one and the second one (problem III) is posed and selected by children. The children are the same children and the lesson was carried out about four months after the first problem I.

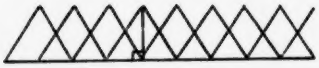
Problem II (Given Problem)

There are 3 square sheets of paper with sides of 12 cm. These are overlapped along the diagonal line as shown in the figure. The non-overlapping part of each side is 4 cm. What is the perimeter of this figure in cm?

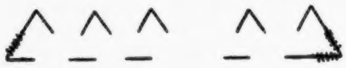


Problem III (Problem Selected)

There are equilaterally triangular sheets of paper with sides of 8 cm as shown in the figure. The perimeter of this figure is 4752 cm. How many sheets of paper are used?

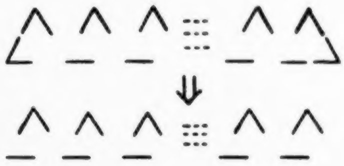


Problem III was taken up because most of the children wanted to solve it. Generally speaking, children tend to choose the difficult problems.



4752 - 4 x 3 = 4740

4740 ÷ (4 x 3) = 395



One of the children's solutions is shown in the above. This child uses the idea of correspondence. It is easy to understand.

I wish to acknowledge the suggestions and help of the Seminar's Japanese members, especially those of Professors Tatsuro Miwa and Shigeru Shimada. Edith Sarra and Robert Easley assisted me with the translation.

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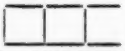
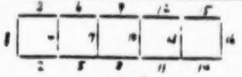
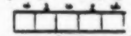
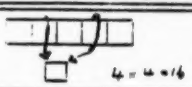
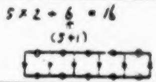
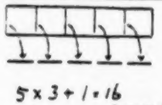
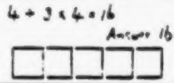
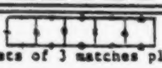
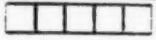
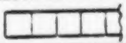
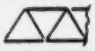
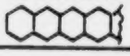
Appendix Record of the Whole Lesson Hour

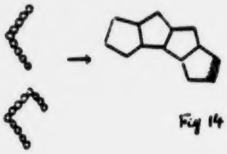
	Teacher	Children
1	Let's begin.	
2	First, we will review the last lesson.	
3	This is the problem we studied the day before yesterday: Squares are made by using matches as shown in figure 1. When the number of squares is 5, how many matches are used?  Fig. 1 You came up with various ways of solving the problem, didn't you? Let's introduce some of them.	
4	One is by Yoshimura-kun. He wrote the number on the line and went on to count 1, 2, 3, ---.	 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 Fig. 2
5	In the same way, Nozaki-kun asked for matches. I gave them to him. He actually arranged 16 matches and made a thorough count..	
6	Though Sonobe-kun found that his method was wrong, he showed it to us. As there are 5 squares and each has 4 sides, 4 times 5 is 20. Twenty is wrong, because he counted double. He explained his mistake by himself.	Wrong answer example As it is 5 squares, I can get an answer by counting the number of matches in a square and the number of squares. Fig 3
7	Third, there were two responses that were the same. There are three separate squares, each with four matches. Between them there are 2 sets of 2 matches, so the total is 16 matches.	 There are 3 sets of 4 matches and 2 sets of 2 matches. Fig 4 Answer 16 matches
8	Then Kosaka-san said we can make a square by combining the matches that are in between and then there are 4 sets of 4 matches.	 Fig. 5
9	Well, Tsunashima-kun gave his opinion. If there are many squares, can we still use this method? We had such a question.	
10	Well, do you remember Tani-kun's method? There are 5 above and 5 below, so 5 times 2. And 6 is 6 standing up. Someone asked him: What is this 6? Tsunashima-kun said something. What did you say? Because 6 is 1 more than 5, 5 plus 1 is easier to understand than 6. If so, we may use this method in other cases.	 Fig 6 If the number of squares is larger, is it still 6?
11	Then there were many other solutions. Ariga-kun's method was to lay down the vertical matches, so there were 3 sets of 5 matches and one remaining match. This is similar to the last method.	 Fig. 7 5 x 3 + 1 = 16 Answer 16 matches

	Teacher	Children
12	There are 3 more examples. Tsuda-kun made a square first, and then the letter "コ" (Note: ko --- from the Japanese syllabary). So the total number of matches is the 4 plus 3 times 4, or 16.	 $4 + 3 \times 4 = 16$ Answer 16 matches Fig. 8
13	Then, Yamashita-san made 5 sets of 3 matches using a reversed letter "コ" plus one remaining match. So, 3 times 5 plus 1, or 16.	 5 sets of 3 matches plus the remaining 1. $3 \times 5 + 1 = 16$ Answer 16 matches Fig. 9
14	And last is Suzuki Hiromi-san. 4 matches make a square and there are 5 squares, so 4 times 5 is twenty. But, the red lines are counted double because they overlap, so subtract 4. As there are 5 squares and the number of vertical matches is 5 plus 1, we have to subtract the left and right ends from 5 plus 1. This would be a better method.	 $4 \times 5 - \underset{\substack{4 \\ (5+1-2)}}{\underline{4}} = 16$ Answer 16 matches Fig. 10
15	We had 8 different ways of solving the problem. You pretty much understand? I checked your responses. There were some good ideas and 2 or 3 I didn't understand.	
16	In the last lesson, we solved only the first problem. In today's lesson, I won't pose a problem, but you will pose it by imitating the first problem. I want you to present the problem you made yourselves and discuss it with each other. As I pass out the sheets, pose a problem and write it out. All right? Get out your pencils.	
17	Let's read it briefly. We could solve this problem in various ways. Let's pose a problem similar to this problem. This is today's lesson. Now, although Ariga-kun asked me "May we pose as many problems as we want?", make only as many problems as the time allows.	
18	Then let's begin. Please work without hesitating though the teachers may walk around the desks. Does anyone have a question? Okay? If there's anything you don't understand, please ask me.	
19		Is it okay if it's only a little bit similar?
	Yes, that's okay.	May it be a problem using matches like this one?
	Yes, matches are okay. Or maybe something else? Something besides matches might be better.	
20		Is it okay if we use triangles or pentagons instead of squares?
	That's a good idea. But if you say your good ideas out loud, others may end up using them. Let's begin.	

	Teacher	Children
21	What do you mean you don't understand? You understand what you have to do, don't you? It's easy to understand if you imitate the first problem.	
		Teacher, may I draw a figure?
	Yes, if you draw a figure, it will make it easy for others to understand your problem.	
		Do we have to write an answer?
22	No need. For now, just the problem.	What if I don't know the answer to my own problem?
	That would be interesting too. So we'll think about it afterward. It's okay. Even if you can't find your own answer, it's okay. As it is interesting, we'll introduce it afterward.(Note: Walking around the desks, the teacher says this to a child. He gives the child posing the problem a sheet and a magic marker to write with.)	
23	It'd be great if you could write two problems or so. All right?	
24	Then look here. When you finish what you are writing now, let me know by putting your pencil on the table. I walked around the desks and had 5 people write their problems. Listen to their problems and if you have made a problem similar to their problems or if you have a question about them, please speak up. Any questions?	
25	Then, Sonobe-kun, please come here and explain.	I changed the first problem a little, and made the following problem by changing to iron sticks. Squares are made by using iron sticks. If the number of squares is 30, how many iron sticks are used?  Fig 11
26	How did he change it? Does anyone understand? How did Sonobe-kun change the first problem?	I changed the matches and the number of squares.
27	I have one question about it, but does anyone else have questions? What is iron sticks? He wrote just iron sticks, though it may be okay as he drew a figure of it. But what is the length of the match sticks? All are the same length. What kind of sticks did Sonobe-kun draw?	

Teacher	Children
28 We drew them by hand. Perhaps we should add something to this. What should you add? What should you add, Ariga-kun?	Constant-length iron sticks.
Oh, is that so? And what about you?	The same length.
Sticks of the same length. You had better add something like that. Constant-length. You know a difficult word, don't you? Did you learn it in fourth grade?	Yes, we did.
29 You had better add such an expression. You had better add the word "constant-length" or "the same length". Well, you changed matches to iron sticks. And you changed 5 to 30. Any questions? Did anyone make a problem similar to this?	
30 Yes, what did Shoji-kun write? Is it similar?	The number is different.
What is your number?	Seventy sticks.
31 Seventy sticks. You have written that you made squares by using 70 matches. Is it really similar to Sonobe-kun's? Did Sonobe-kun use 70 matches? You know the number of squares is 30. How many people changed the number of squares?	(Note: About 10 children raise their hands.)
A few.	
32 Yes, this type of problem was more common.	
Tani-kun, please explain this. Listen to his idea.	Squares are made by matches in the first problem. I made the problem by changing squares to equilateral triangles. Any questions?
Would you read it?	(Note: Tani-kun reads the problem below.)
	Equilateral triangles are made by using matches as shown in the figure. When the number of equilateral triangles is 15, how many matches are used?  Fig. 12
33 Did you only change squares to equilateral triangles?	I also changed the number.
You changed the number too. When I walked around the desks, I found many problems like this one. Raise your hand if you changed squares to triangles.	(Note: Many children raise their hands.)
Oh, too many.	
Well, how many people changed squares to geometrical figures other than triangles?	

	Teacher	Children
34	What figures did you make? Endo-kun, come up and put your paper on the blackboard.	
35		Well, I changed squares in the first problem to regular hexagons,and I changed the number of squares is 5 to the number of figures is 1011.
36	Please, read the problem.	(Note: Endo-kun reads the problem below.)
		Matches are arranged as shown in the figure. When the number of regular hexagons is 1011, how many matches are used? (the length of matches is all the same)  Fig. 13
	Can you solve it?	It's solvable if I compute it.
	How did you compute it?	Maybe I can.
	I think you can.	
	Both of you changed the figure. You changed the figure. There are two types of children, those who changed squares to equilateral triangles, and squares to regular hexagons.	(Note: The teacher writes the following things on the blackboard.)
		figure square → equilateral triangle regular hexagon
37	Did anyone pose the problem by changing to other figures besides the hexagon?	
38	What did Suzuki-san do?	Should I read the problem?
	Please read your problem. Come here, and write it on the blackboard with chalk.	
	Would you read your problem without writing it?	I want to make four pentagons with 5 beads per side. How many beads are used? (Note: She begins to write the problem.)
	How did you make it? Please draw your figure.	
39	Suzuki-san said the figure was a pentagon. Did anyone else make pentagons? Please draw it. How do you connect the beads? Please draw the figure. Please draw it here.	
40	By hand is okay.	

	Teacher	Children
41	Triangle, hexagon, pentagon. Did anyone make other figures? Yes, Kosaka-san.	Rectangular solid.
	Rectangular solid? Did you draw it? That's interesting. You can't really do that with matches.	
42	Tsunashima-kun?	Rectangle.
	Can you make a rectangle? Draw your figure.	
43	Suzuki-kun? I had you write something. Put it on the blackboard. (Note: To Tsunashima-kun) Write your idea above Suzuki-san's response.	(Note: Suzuki-san drew the left figure first, and next changed the right figure.)  Fig 14
44	(Note: Suzuki-kun brings his sheet.) Okay, lay it over there.	
45	You just fill in one side of beads with yellow chalk, we can understand.	
46	Please explain how it all works, Suzuki-san. Yes, Tsunashima-kun. Do you get it? She drew her figures on the board like this, now listen to the problem.	I want to make four pentagons with 5 beads per side. How many beads are used?
	How did you change it?	Well, the figure is written here, and please consider this as a bead. Pentagons are made by arranging 5 beads this way and that.
47	She made such a problem. She didn't use matches, but the figure is a pentagon. Thanks. Any questions? Ariga-kun, okay?	Yes. (Note: He reads the following sentence.)

A vertical figure of regular pentagons is made by using matches. When the number of regular pentagons is 726, how many matches are used?

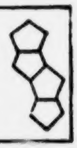
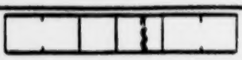


Fig 15

Regular pentagons are connected like this. It is different from the first problem, because in his problem, the figure is zigzag while the first one is made horizontally. Thanks. Any questions?

	Teacher	Children
48	Then Tsunashima-kun. What is your problem?	 Fig.16 Rectangles and squares are made by using matches in the figure. When the number of rectangles and squares altogether is 1111, how many matches are used? (One side of the rectangle is two times that of the square.)
49	I don't understand the meaning of the figure.	I know. Rectangle, square, rectangle and square. (Note: Tsunashima-kun calls on Ariga-kun, who has raised his hand.) If one side is doubled, is each side of the rectangle doubled? Are both width and height doubled?
50	What Tsunashima-kun just said at the end of his problem was that one side of the rectangle is two times that of the square. Ariga-kun is asking about that. A rectangle has four sides you know. Only the width, isn't it? Your figure shows that, but your description is a little vague.	Only the width. (Note: Only the width is doubled.)
51	Well, one more person, Suzuki-kun, came up with a different way of posing the problem. Suzuki-kun, this is yours? (Note: See No. 44.)	Yes. I almost completely changed the problem. And this is the problem.

Parallelograms are made by using pencils of the same length as shown in the figure below. When the number of pencils is 37, how many parallelograms can be made and how many pencils will remain?

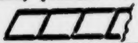


Fig.17

	Teacher	Children
	How did you change it? You said you almost completely changed it.	Yes. I changed matches to pencils and squares to parallelograms. And instead of asking how many matches make squares, I asked how many parallelograms can be made with 37 pencils and how many pencils will remain.
52	It seems a little difficult, but any questions?	What if there is no remainder? It's okay.
53		Ariga-kun.

	Teacher	Children
		Finally, even if you use parallelograms, the length of the edges are the same. So I think it is almost the same as using squares.
		But I thought that I had better change it.
54	Is this the same as a square? As the number of edges is four and the length of the pencils is the same, it is a square. Is that okay? What is a square?	In a square all the angles have to be 90 degrees, so that is not a square.
	Then what is it? What is it called? The sides are the same length, but it's slanting.	It's a rhombus. The pencils are the same length, so it's a rhombus.
	Calling it a parallelogram is okay. It must be a parallelogram. But, as it is described as having sides of the same length, calling it a rhombus is better, according to this opinion.	
55	Then, when you posed this problem you asked how many pencils would remain. Did you pose the problem knowing what the remainder would be?	I thought maybe there would be a remainder.
	Maybe? Okay, let's consider it later.	Zero if there is no remainder.
	Your opinion is that if there is no remainder then the remainder is zero. It might be good to add that.	
56	Suzuki-kun's problem is very different from the first problem. The first problem describes the number of squares, but in his problem the number of pencils is already determined.	
	So can we apply the previous calculation?	We can't.
	What kind of calculation then? It may not flash across your mind immediately. Multiplication?	Division.
	It may require division.	
57	How about Yamashita-san's problem, how does it resemble Suzuki-kun's? Let's think about it a little. How are they alike?	(Note: Yamashita-san comes up to the front and reads the problem.)

There are 27 sticks. Triangles, some triangles are made by using these sticks. How many triangles can be made?

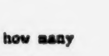


Fig.11

	Teacher	Children
58	Any questions? Ask them their opinions.	You say triangles, but is any kind of triangle okay? Is it okay if there are all different kinds of triangles?
	If you look at the figure, it doesn't look like it's okay. So how should we fix it? How would you fix it for her?	If you changed "triangles" to "equilateral triangles" it would be okay.
59	They're all the same length, aren't they? So, it would be good to say "equilateral", wouldn't it? (Note: To Yamashita-san) Is that what you meant?	Yeah.
	So, how would you re-state the problem?	I think you don't need the part about "some triangles".
60	If you do that, then how does it sound?	There are 27 sticks. Equilateral triangles are made by using these sticks. How many triangles can be made?
61	Isn't this statement still a little vague? Equilateral triangles, some equilateral triangles are made by using these sticks. You put "equilateral" on everything, so doesn't it sound a little funny, Tani-kun? If you get rid of some of that, it will sound right.	Yeah.
62	Please read it. (Note: As he reads, the bell rings.)	

Ther are 27 sticks. Equilateral triangles are made by using these sticks. How many equilateral triangles can be made?

Fig. 19

	Teacher	Children
	That's it, isn't it? Yes, thanks. You can sit down.	
63	What is the common point between Suzuki-kun's and Yamashita-san's problems? Can you find it, Sato-san?	Since Yamashita-san asked how many equilateral triangles can be made, they both first determined the number of matches, and then asked how many equilateral triangles or rhombuses can be made?
64	That's right. Tano-kun?	What would happen if the sticks are too big?
	Yes, here's another opinion. If the sticks are too big, what would happen? What do you mean by that?	In short, we can't arrange them.

	Teacher	Children
65	Oh, you mean they can't be arranged well. But the sticks are like matches, right?	
	Yes?	In Yamashita-san's problem, if there is a remainder, what do you do?
66	Yamashita-san, how did you come up with the number 27?	Because it's a divisible number.
	What number is it divisible by?	3.
	Did you decide on 27 because it's divisible by 3? As Sato-san just told us, in both Yamashita-san's problem and Suzuki-kun's problem, they first determine the number of matches and then ask how many figures can be made. Is this a different type of problem?	
67	Suzuki-kun, when you thought of this problem, why did you use this number, for it is not divisible by 4. Did you think that if it is not divisible by 4, it would have a remainder? Yamashita-san thought of 27 since it is divisible by 3. Is this method all right for posing this kind of problem? Is it all right? Who thinks it's all right?	(Note: Not much response)
	Did anybody else make this kind of problem?	(Note: Many respond)
	Many people made problems in which there is a certain number of matches and you ask how many figures can be made using them.	
68	What did Okutomo-san do? Listen to her idea.	Triangles are made by using matches. When the number of matches is 135, how many triangles can be made?
69	Now the problem is, how did you think of the number 135?	First I wrote a large triangle, and added many small triangles, and counted the number of edges.
	Oh, is that how? I asked without looking at what you did. (Note: Looking at her sheet) Okutomo-san didn't arrange the triangles in a line. She thought of a big triangle and filled it with little triangles. In her case, though, she started from the top of the triangle and worked down. So that's different from the other problems. You thought of the number by counting up the number of sides.	
70	So, then, returning to the previous problem, with 27. In Ariga-kun's question a few minutes ago, he wondered if the answer to the problem has no remainder.	
	What do you think?	In Yamashita-san's figure, I think the number is not divisible by 3, because two edges overlap.

Teacher	Children
What do you mean? I don't understand the meaning. Please come up to the front and explain.	Because, Between one triangle and the next. Isn't there an overlap? If there is one, it's not divisible by 3.
Really? In Yamashita-san's problem, first 3 sticks, then 3 more sticks. Shall I draw it again here? If so, it is divisible by 3. Okay? (Note: The teacher draws: △▽△)	
Kaneko-san?	I agree.
Tago-kun?	It doesn't matter if it's 28 or 30 or any number because that is not related to the problem.
Why?	Because it's not a problem in which it must be evenly divided.
71 As it is a problem about how many figures can be made, it's okay if it has a remainder, right? It's no problem whether it has a remainder or not. So the number is not related to the problem so much. But Yamashita-san said she thought of 27 because it was divisible by 3. The person who made the problem thought that there was no remainder in the answer. But it may be like the explanation Ariga-kun gave. However, as a problem, it doesn't matter if it has a remainder because the question is, how many figures can be made, as Tago-kun explained. So the problem is okay.	
72 Then, time is up, but the first method by Sonobe-kun is increasing the number of squares and the methods by Tani-kun, Endo-kun, Kaneko-san, Suzuki-san and Tsunashima-kun involve changing the figures. Of course, the number is also changed. These two interesting problems are different from the others. Their problems give the number of matches and ask how many figures can be made. Each problem belongs to one of three types. In the first type, the number is changed, that is, the number of squares. In the second type, the figure is changed. The third type of problem reverses these problems.	
What problem would you want to solve if you were to solve one of these problems?	Endo-kun's problem.
And you?	Endo-kun's problem.

(The End)

Discussion of Professor Hashimoto's Paper

Travers: Thank you, Yoshihiko, and thank you especially for bringing the video-tape which helps to give us a good feeling for the Japanese classroom. Are there questions or comments people would like to raise?

Kantowski: This is an excellent example of problem solving in the elementary school. These activities are related to what Polya calls looking back activities, where students look back at the solution to their problem and then try to find another problem. I think this is a valuable exercise in teaching students about problem solving. I was very pleased to see the tape and I think you should be commended for bringing this to our attention and giving good examples of the kind of problem solving that is important.

Hashimoto: Thank you.

Silver: I'd like to know how typical this teacher is of teachers at the upper elementary level in Japanese schools. Is this an average class in the Japanese schools, or is this unusual?

Hashimoto: I think the teacher is above average but the children are average because this school is an ordinary public school located in Tokyo.

Silver: In terms of the nature of the activity of the classroom, is this a typical mathematics lesson?

Hashimoto: In my opinion the theme of this lesson is experimental and not common in Japan, but the style of interaction between teacher and children is typical of Japanese elementary schools. This kind of lesson has been promoted since 1978 as I wrote in my paper. We have about 100 examples from 1st grade to 12th grade.

Travers: I have a marginally related question. At the end of your paper you raised the issue of the philosophy of mathematics teaching and one example of teaching in Japanese classrooms. The kind of activity that goes on has impressed me since you mentioned it when you were with us at Illinois, and that is the practice that is encouraged, I understand, that teachers keep records of their teaching activities that might be of an experimental nature. These records are almost in the form of an action-research kind of approach to teaching, with the teachers being encouraged to keep an account of what goes on. Would you want to comment a little more, perhaps in the context of what we have seen?

Hashimoto: Yes, we have many videotapes and have published some books and presented many papers about our research. When such teaching is carried out, I think it is important that other teachers observe the lesson and that it be recorded carefully. After finishing the lesson, professors or math education researchers discuss the lesson with classroom teachers. When I stayed at the University of Illinois I felt that the collaboration between university professors and classroom teachers was not large, but the collaboration between university professors and graduate students is very close. The role of professors seems to be different in both countries.

Clarkson: I want to also tell Professor Hashimoto how much I enjoyed the talk. One of the things that I am finding today is that there are at least five different things that are written in Japanese that I would like to read, and I'm feeling very much at a disadvantage. I was wondering whether we could get some of the papers that are referenced translated into English?

Becker: Perhaps. Tomorrow we will get some books from the Japanese. I don't know whether it's exactly what you are referring to but that's something that we should look at.

Clarkson: I was especially referring to one of Professor Hashimoto's papers, but another reference that I was looking at was Professor Shimada's lessons using open-ended problems in mathematics teaching. They are also in Japanese; there are several others like that. Maybe at a future time we could get them.

Shimada: Two English publications will be shown tomorrow. We have some samples. But for those books written in Japanese only, we have no means to translate them. Translation would cost too much.

Miwa: In Japan, mathematics teachers in elementary and secondary schools and university mathematics educators have an organization called the Japan Society of Mathematical Education (JSME). Probably it corresponds to your NCTM. JSME publishes a monthly journal, which includes five or so articles by teachers or mathematics educators. In addition, JSME has an annual meeting in August every year. Then we have 150-160 articles in the elementary school section and, in total, over 400 articles which are published in a special issue of the journal. Of course they are written in Japanese. I regret that we have no journal on mathematics education written in English. I hope, in the near future, we can get support and we can publish the journal in English - it contains some interesting articles from Japan.

Now, Professor Hashimoto's article, which shows his basic philosophy and examples of his teaching, is contained in the Proceedings of ICMI-JSME Regional Conference. It is edited by Mr. Sawada. Tomorrow morning the Japanese delegation will give a copy to the U.S. delegation along with two other books.

Becker: Professor Hashimoto, you made reference at the end to the philosophy of the teacher. Where does that philosophy come from? Is it self-generated by the teacher, or does it come from the teacher training program?

Hashimoto: That's a difficult question to answer, but I think both. Let me give you an example. I have done work with the teacher referred to in my paper since 1972. In our project math education researchers and professors make a team with classroom teachers. We discuss our project with each other and convey the philosophy of such kind of mathematics teaching to the teachers. They will understand it gradually. Of course, teachers themselves have to think through the classroom lesson and have to make an effort, and they also have to be eager to do such kind of teaching. I think both are important. This is one example.

Becker: One more question. If a teacher decides that he or she would like to teach using this approach in the classroom, is there a lot of freedom to do it?

Hashimoto: Yes, they have the freedom.

Becker: What about the preparation for tests?

Shimada: After the 2nd World War, the entrance examination to secondary school from elementary school was abolished for public schools because compulsory school education was changed from 6 years to 9 years. Therefore, elementary school teachers became free from examination pressures. But in city areas and high society, parents want to send their children to private or national junior secondary schools. In that case, pressure from examinations still prevails. However, the school you saw on the VTR screen today is not so much disturbed by such examination pressures.

Travis: Earlier, Dr. Wilson talked about the An Agenda For Action published by the National Council of Teachers of Mathematics to help focus teachers' emphasis on problem solving and to help make it an integral part of classroom teaching. Do the Japanese teachers have a similar document to focus their concern on problem solving and to make them aware of the need to integrate problem solving into the classroom?

Shimada: The Japan Society of Mathematical Education is now preparing some kind of Agenda for Action. I am not a director of the Society but an outsider of the Board of Directors. But I heard that news from the President. As for the importance of problem solving activities in mathematics education, the Japanese situation is not so much like that of the United States. Many teachers emphasize problem solving and they have had that experience beginning in the period 1940-50 - I made reference to this in my paper. During that period the curriculum was wholly based on problem solving, but it did not succeed as expected. It was claimed that learning based on problem solving activities led to a sacrifice of the development of mathematical knowledge.

And at present, some persons put much more emphasis on mathematical understanding of basic concepts than on problem solving. They say that if a child can understand them, that is, the essence of mathematics, then the child will be able to solve problems when faced with real ones.

Furthermore, I would like to add a few points in considering real problems in school. In the classroom situation, the school has a duty to protect its students from any danger such as mental, physical, or economical. If some problem which seems challenging and suitable for students' level of ability has a risk of such a danger in the process of solution, then it cannot be used in teaching. For example, gathering data for traffic in dangerous street corners, or investing money in a real stock market. So there are many who think that realistic problems may not be appropriate for school teaching and must be modified so as to get rid of possible harms; in other words, that real-like-problems are better than real problems.

In fact, there are many teachers who emphasize problem solving, but at the same time, there are many who think that it is adopted as a means of promoting mathematical ability and its objective is understanding of the essence of mathematical problem solving but not mathematics for its own sake.

Travers: I would like to raise another issue which may be related to the document on which the Japanese Society for Mathematical Education is working. It has to do with the relationship between the teacher of mathematics in the secondary school and teachers of other subject matters. We talked about real problems and yet it seems to me that often the mathematics teacher is isolated for whatever reason from the other subjects. Is this a concern in Japan and is anybody doing anything about this?

Shimada: I do not know what will be included in the document that JSME is working on. As for the relationship between the teachers of mathematics and teachers of other subjects in the secondary schools, the situation is just as you pointed out.

It is usual in Japan that mathematics teachers are isolated from teachers of other subjects with respect to teaching of mathematics.

Becker: Reference was made this morning to the expectations of parents and teachers of the students in mathematics. We have seen now some examples of problems which I would regard, at the grade level that they are used, as challenging, maybe somewhat difficult problems. And I have also heard from other Japanese, though I don't know how true it is, that it's a cultural characteristic in Japan that learning mathematics is part of improving oneself as a human being. I wonder if some of the Japanese delegates could react to this.

Miwa: One comment about the question. Generally speaking, in the principle of school education, all subjects should contribute to the character formation of students. Of course, mathematics is one of those subjects. If it is taught only for the sake of mathematics, it is of no use. This is based on the principle above. Actually, people often see mathematics as important because it is the key subject in entrance examination, other than its value in science and technology only for a very limited elite.

Here I would like to refer to the layman's inconsistent dual views on mathematics in Japan. One is that people view mathematics as very precise and correct. For example, they are very confident of its importance because of the entrance examinations. Another is that people often say that mathematics is mathematics and it is not real, and that in the real world one plus one is not always equal to two.

Wilson: I want to pose a question that seems to cut across several sessions. All the discussions start with or seem to imply an assumption of absolute acceptance of the goal of problem solving and problem solving goals in our curriculum. I raise the question, hoping no answer will be given at the moment, is it time for us to also reassess whether we, or to what extent, really hold that goal?

Travers: This is something for us to think about this evening. Again, Yoshihiko, thank you. We are adjourned.

SESSION 3

Professor Sawada's Paper

Wilson: To open this session this morning, our speaker will be Mr. Toshio Sawada from the National Institute for Educational Research (NIER). I first met Mr. Sawada during my visit to Japan in 1978. We were very fortunate to have a return visit by him to the University of Georgia in 1980 and we have had the opportunity to exchange information with one another over the years. One of the reasons I went to Japan in 1978 was to attend work meetings in connection with the IEA study and, of course, Mr. Sawada has been deeply involved in that. His paper this morning will be a presentation on Japan's results from the IEA studies.

Sawada: Thank you, Mr. Chairman.

COMPARISONS OF ACHIEVEMENT IN PROBLEM SOLVING ON SIMS BETWEEN THE UNITED STATES AND JAPAN

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0. Introduction

In 1980-82, the International Association for the Evaluation of Educational Achievement (IEA) conducted the Second International Mathematics Study (SIMS) in the schools of 20 countries, including the United States and Japan.

Making comparisons of students' achievement between the United States and Japan is a difficult task at best. Both countries differ widely on a number of social, cultural, economic, and political factors that may have an impact on the content of the mathematics curriculum, on the way in which mathematics is taught, and on students' learning of mathematics.

The main purpose of this paper is to compare mathematics achievement, especially achievement in problem solving between the United States and Japan. The data used in this paper came from the results of SIMS for Population A (see National reports and others).

The international definition for Population A was all students enrolled in the grade where the majority of students have reached the age of 13 by the middle of the school year. In the United States that population would be eighth-grade students, and in Japan, because of difference in the nature of the school year, that population would be seventh-grade students whose average age was slightly less than 13 years by the middle of the school year. These data were collected from approximately 7,000 eighth-grade students and 280 teachers in the United States, while Japanese data were collected from actually 8,091 seventh-grade students and 212 teachers.

1. Teaching and Learning Mathematics

1.1 Goals in Teaching Mathematics

Teachers were asked to rate the relative emphasis that should be given to each of nine objectives in mathematics instruction. Table 1 presents the results of these responses (percentage of teachers who rated as 'very important') for Population A teachers in the U.S. and Japan.

Table 1 Goals in Teaching Mathematics

	Objectives	USA	JPN
a.	Understand the logical structure of mathematics	30%	13%
b.	Understand the nature of proof	12	10
c.	Become interested in mathematics	45	65
d.	Know mathematical facts, principles and algorithms	55	20
e.	Develop an attitude of inquiry	39	32
f.	Develop an awareness of the importance of mathematics in everyday life	61	18
g.	Perform computations with speed and accuracy	58	48
h.	Develop an awareness of the importance of mathematics in the basic and applied sciences	20	14
i.	Develop a systematic approach to solving problems	63	34

The U.S. teachers reported that their most important goals were developing a systematic approach to solving problems (rated as 'very important' by 63% of the teachers) and developing an awareness of the importance of mathematics in everyday life (61%). These were followed by computing quickly and accurately (58%) and knowing mathematical facts, principles and algorithms (55%).

Japanese teachers reported that their most important goals were becoming interested in mathematics (65%) followed by performing computations with speed and accuracy (48%). Problem solving was in third place (34%). Both countries gave their lowest emphasis rating to the goal of understanding the nature of proof for Population A level students.

1.2 Teaching Activities

Teachers were asked to estimate the number of minutes they devoted in a typical week to the following teaching activities. Number of minutes (in average) in a typical week spent by Population A teachers in the U.S. and Japan are as follows:

		USA	JPN
A.	Explaining new mathematical content	98 mins.	78 mins.
B.	Reviewing or revising content	59	41
C.	Routine administrative tasks	28	16

In both countries, most of the time is spent in explaining new content to the class to more than one student at a time.

Teachers were also asked to estimate the average time spent by students in the target class during a typical week on the following class activities:

		USA	JPN
D.	Taking mathematical tests	41 mins.	20 mins.
E.	Seatwork	100	58
F.	Listening to lectures or explanations	85	76
G.	Working in small groups	33	15

According to teachers' estimates for typical week, the proportions of taking test and working in small groups are 9% and 16% of total students' time in both countries. Japanese students spent relatively more time on listening - discussion and less time on working in small groups than the U.S. students.

Then both teachers and students were asked to respond on the 'mathematics in school' scale. Each item describes either a mathematical topic or an activity that were believed to be almost universally part of the mathematics classroom. The following four items were common to both teachers and students and their responses to 'very important' are as follows:

	Teacher		Student	
	USA	JPN	USA	JPN
	%	%	%	%
H. Checking an answer to a problem by going back over it	53	62	27	48
I. Memorizing rules and formulas	32	35	38	63
J. Solving word problems	61	43	20	51
K. Estimating answers to problems	22	29	20	18

The U.S. teachers reported that their very important items were "solving word problems" (61%) and "checking an answer to---" (53%), while Japanese teachers was to "checking an answer---" (62%). On the other hand, Japanese students show a relatively higher percentage than the U.S. students, except item K.

1.3 Teaching Style

In some mathematics classes, all students are expected to do the same work. In others, the teacher adjusts and selects the work to suit the ability and experience of individual students or groups of students in the class, by requiring them to do different questions or exercises. Teachers were asked to estimate the percentage of target class time in a typical week devoted to each of the following.

Table 2 Percentage of Target Class Time

	USA		JPN	
Objective	mean	S.D.	mean	S.D.
	%		%	
a. Whole class working together as a single group	48	26.0	69	18.3
b. Small group instruction	19	22.5	10	10.2
c. All students working individually	32	22.0	19	15.3

The average percentage of whole class working together as a single group in Japan is 69%, a relatively high percentage, whereas small group instruction or all students working individually present relatively lower than the U.S. responses.

In the classroom in Japan, generally the teacher-centered instruction is adopted by giving the same tasks to all the students in it.

On the other hand, school principals were asked as to whether setting or streaming took place or not in their schools. The proportions of setting or streaming in mathematics classes is 77% in the U.S. and only 2% in Japan. It seems that most teachers of Population A classes in Japan clearly grasp the realities of mathematical ability range of students in their classes, but it is the general practice not to stream classes. This may be thought of resulting from teachers' negative belief in the value of streaming.

2. Intended, Implemented and Attained Curriculum

SIMS has three analyses of curriculum. At the level of the educational system, there is the intended curriculum, the collection of intended outcomes, supplemented by course outline, official syllabi, and textbooks which prescribe what is intended to be taught in mathematics. In Japan, the basic framework including the objectives and standard teaching content is outlined in the "Course of Study" issued by the Ministry for each of the three school levels: elementary (Grade 1-6), lower secondary (Grade 7-9) and upper secondary (Grade 10-12). Textbooks to be used in schools are those published by commercial publishers after the authorization by the Minister of Education.

At the second level, the level of the classroom, there is the implemented curriculum. Teachers may exercise their own judgment in translating curriculum guides or adopted textbooks into a program for their class. Thus, the implemented curriculum will reflect the personal preferences and biases of teachers and the coverage of textbooks used, as well as the size and composition of the classes in which mathematics is taught. It is clear that the implemented curriculum need not bear so strong a resemblance to the intended curriculum, and that it is the implemented curriculum which finally determines the student's opportunity to learn mathematics.

Finally, the information and skills mastered by the students, as demonstrated in tests and questionnaires, makes up the attained or achieved mathematics curriculum. The extent to which the three curricula correspond to each other is an important concern of the IEA Study.

2.1 Intended Content Coverage

In SIMS, for each item the national committees were asked to give a rating of 2 if the item was highly appropriate to the national mathematics curriculum, 1 if the item was acceptable in terms of the curriculum, and 0 if the item measured mathematics content not in the curriculum. After categories 1 and 2 had been combined, these ratings formed the basis for the variable termed the index of intended content coverage; that is, the percentage of items in the SIMS test regarded as acceptable or highly appropriate. The index represents the extent to which an item was seen to be in the official curriculum, or not.

The curriculum analysis was based upon a content-behavior grid. The four levels of behavior were computation, comprehension, application and analysis, and the area of content for Population A included arithmetic, algebra, geometry, probability and statistics and measurement. Also, some items were classified to non-verbal (computation problem) and verbal (e.g., problem solving) category.

The description of behavior levels was based on a chapter by J. Wilson in B. S. Bloom's book. Computation was taken to mean ability to recall facts, to use mathematics terminology and carry out algorithms. Comprehension included the ability to recognize concepts, mathematical principles and rules, to transform problem elements from one mode to another, to follow a line of reasoning, to read and interpret a problem and to generalize. Application included the ability to solve routine problems, to make comparisons, to analyze data, and to recognize patterns, isomorphisms, and symmetries. Analysis included the ability to solve non-routine problems, to discover relationships, to construct proofs, to criticize proofs, and to formulate and validate generalizations. More interesting for comparison are the results of students' achievement in problem solving, which are included in the application and analysis levels.

A summary of the indices of intended content coverage (APPR) is given in Table 3. Total averages of 157 common items are 82% in the U.S. and 94% in Japan. Topic subtests rated over 90% in the U.S. are arithmetic, statistics, measurement and verbal areas, and all areas except geometry in Japan. Geometry and algebra show relatively low intended content coverage in the U.S.

Table 3 Means on Selected Variables Related to the Intended and Implemented Curriculum by Topic Subtest: Population A

Topic Subtest	Number of Items	Intended Curriculum		Implemented Curriculum			
		APPR(%)		TOTL(%)		TESA(%)	
		USA	JPN	USA	JPN	USA	JPN
Total	157	82	94	69	77	45	51
Arithmetic	46	100	94	85	85	54	52
Algebra	30	63	93	70	83	43	52
Geometry	39	59	87	46	52	32	41
Statistics	18	94	100	72	76	49	54
Measurement	24	100	100	75	95	45	64
Computation	53	79	94	77	79	52	56
Comprehension	51	82	90	70	70	45	48
Application	45	82	96	64	81	38	51
Analysis	8	88	100	47	71	30	45
Non-verbal	15	73	100	87	92	60	63
Verbal	16	94	100	81	97	47	56

Note: APPR: Item appropriateness index
TOTL: Teacher "opportunity-to-learn" index
TESA: Teacher estimate of student achievement

2.2 Implemented Content Coverage

One of the implemented content coverage indices is computed from item level judgements by classroom teachers of whether students in their target class had an opportunity to learn the mathematics skills and content tested by the item. Teachers were asked for each item whether this opportunity to learn had occurred in this year, in a prior year, or never. Table 3 shows "teacher opportunity to learn" index (TOTL) by topic subtest.

Another one of the implemented coverage indices, namely teacher estimate of student achievement, was obtained by asking teachers to estimate what percentage of the students in the target class would get each item correct. This index was also designed to detect whether teachers were able to grasp the reality of students from the likely performance of the same students on the test. Teacher estimate of student achievement (TESA) also is given in Table 3, by topic subtest.

In general, the intended coverage index is greater than the implemented coverage index. One of reasons that intention runs ahead of implementation would be that the official curriculum developers may be overly-optimistic about what teachers are able to cover in their courses.

2.3 Differences between Attained and Implemented Curriculum

Table 4 presents a series of attained curriculum and differences between indices of implemented and achieved content coverage. The differences in the table might be regarded as a crude measure of the efficiency of the learning process in the classrooms of both countries.

Table 4 Means* and Differences** between Indices of Implemented Content Coverage and Student Test Score by Subtest: Population A

Topic Subtest	Number of Items	Attained Curriculum TEST(%)		Implemented - Attained Curriculum			
				TOTL-TEST		TESA-TEST	
		USA	JPN	USA	JPN	USA	JPN
Total	157	45	62	24	15	0	-11
Arithmetic	46	51	60	34	25	3	-8
Algebra	30	42	60	28	23	1	-8
Geometry	39	38	58	8	-6	-6	-17
Statistics	24	41	69	34	26	4	-5
Measurement	24	41	69	34	26	4	-5
Computation	53	50	66	27	13	2	-10
Comprehension	54	46	57	24	14	-1	-9
Application	45	42	64	22	17	-4	-13
Analysis	8	32	60	15	11	-2	-13
Non-verbal	15	51	70	36	22	9	-7
Verbal	16	48	66	33	21	-1	-10

* Average percentage of students correct responses by each subtest

** These differences have been calculated by subtracting each subtest score (TEST) from the corresponding teacher opportunity to learn (TOTL), from the corresponding teacher estimates of student achievement (TESA).

In both countries, the differences between indices of teacher opportunity to learn and test score in content area show small values less than 15 percentage in geometry and statistics, and values well in excess of 20 percentage in arithmetic, algebra and measurement. In Japan, the difference in geometry is -6 percentage. Japanese teachers were not fully aware of opportunity to learn and under-estimated the performance of their students.

The differences between teacher estimate of student achievement and test score are smaller than the differences between teacher opportunity to learn and test score. For Japan these are all negative because of teachers under-estimating the performance of their students.

3. Achievement in Mathematics

3.1 Mean Score in Topic Subtests

Students were tested in the following five content topics: arithmetic, algebra, geometry, statistics and measurement. Each item also was divided into four levels of increasing cognitive complexity: computation, comprehension, application and analysis. The application behavioral level required the solution of routine problems, making comparisons, recognizing patterns and analysing data, while the highest level of all, analysis behavioral level, was defined to include the solution of non-routine problems, discovering relationships and formulating generalizations, including items of problem solving. And some items could be classified according to their non-verbal and verbal nature.

Topic areas show in the column of attained curriculum in Table 4. Total mean score in Japan is 60%. The scores in statistics, measurement and non-verbal show relatively high percentages among topic areas, whereas scores in geometry and comprehension are relatively low. On the other hand, the total mean score in the U.S. is 45%. The scores in arithmetic, statistics, computation and non-verbal show relatively high percentage among topic areas, whereas geometry and analysis are relatively low. In short, content topics in geometry show relatively low score in both countries.

3.2 Item Difficulty

Table 5 shows the correlation table of mean p-value (percentage of correct answers of 157 items) in both countries.

Table 5 Distribution of Item P-value (157 Items for Pop. A)

J P N

P-value	0-	10-	20-	30-	40-	50-	60-	70-	80-	90-	T(i)
0-				1							1
10-	1	1		3	2	2					9
20-		1	3	5	4	4	2	2			21
U 30-	1		1	2	6	4	9	5		1	29
S 40-	1	1		1	3	3	13	8	3		33
A 50-				1	4	3	7	10	11		36
60-				1		1	2	4	3	2	13
70-						2	1	6	1		10
80-	r=0.63								5		5
90-											0
T(j)	3	3	4	14	19	17	35	30	28	4	157

Table 5 shows that there are comparatively small numbers of either very low or very high p-values. The mean p-value is 45.3% for the U.S. and 62.1% for Japan, and the standard deviation is 17.1 for the U.S. and 20.1 for Japan. The correlation coefficient between p-value of the U.S. and those of Japan is 0.63.

It is believed that as the opportunity to learn becomes greater, then the p-values on the items would increase. The correlation coefficient between p-value and teacher opportunity to learn is 0.46 for the U.S. and 0.28 for Japan. The correlation coefficient between p-value and teacher estimate of student achievement is 0.66 for the U.S. and 0.45 for Japan. It seems that the correlations in the U.S. are greater than those in Japan.

According to this table, the number of items with p-value less than 20% are 10 for the U.S. and 6 for Japan. For all of these items, Japanese teachers said that the content had not yet been learned (TOTL under 5%), but for 4 of these items, the U.S. teachers reported that the content had been learned (TOTL more than 70%). The p-value of the item 173 (problem on vectors) was very low in both countries, and also had low intended coverage (TOTL) for both countries. On the other hand, five of the relatively easy items for both countries (p-value 80% more than) fall under the content topics of arithmetic, statistics, and measurement.

3.3 Changes in Achievement since the 1964 Mathematics Study (FIMS)

Thirty-five items from the First International Mathematics Study (FIMS) in 1964 were included in the Second International Mathematics Study (SIMS) for Population A. Each item can be classified into verbal and non-verbal (computational) items according to the criterion adopted in FIMS. The following table 6 presents student achievement between FIMS and SIMS by each topic subtest.

Table 6 Changes in Achievement on FIMS/SIMS

Topic	Number	USA		JPN	
Subtest	of Item	FIMS	SIMS	FIMS	SIMS
Total	35	48%	45%	65%	64%
Arithmetic	14	55	50	65	60
Algebra	9	42	42	55	61
Geometry	5	40	33	67	68
Statistics	5	50	55	73	74
Measurement	2	35	37	73	74
Non-verbal	21	48	49	62	63
Verbal	14	48	41	70	66

For the U.S., the mean p-values for arithmetic, geometry and verbal areas in SIMS are lower than those of FIMS, whereas statistics is higher than before. As for Japan, the mean p-values for arithmetic and verbal in the present study are lower than those of the previous study, whereas algebra is higher than before. As a whole, the topics of arithmetic and verbal (most of problem solving) in SIMS are relatively lower than those of FIMS. These results may reflect the fact that less emphasis in teaching is being given to arithmetic in both countries than was the case in the previous study.

For example, one of the most dramatic decreases occurred on item 078 (IEA code number), shown below.

(Item 078) A runner ran 3,000 meters in exactly 8 minutes. What was his average speed in meters per second?

A 3.75 B* 6.25 C 16.0 D 37.5 E 62.5

(* correct answer)

P-value	FIMS	SIMS	difference
USA	43	19	-24
JPN	51	37	-14

The p-value in the U.S. went from 43% to 19% and in Japan, from 51% to 37%. These differences are very large and the scores fell dramatically.

4. Achievement in Problem Solving

Concerning students' achievement in problem solving, it is of interest to make comparison of achievement for some items in word problems, and in the analysis level in SIMS.

4.1 Achievement in Word Problem

In SIMS, there were 175 items common to both countries. Among them, sixteen were verbal problems; ten involved in Arithmetic; four Algebra; one Geometry; one Measurement. For word problems, the mean p-value is 48% in the U.S. and 64% in Japan as shown in the table 4 (in page 128). The differences between teacher estimate of student achievement and the mean p-values of word problems are -1% in the U.S. and -10% in Japan. The Japanese teachers were not fully aware of their under-estimated student achievement for this content.

The following problems exhibited the lowest percentage of correct answers in spite of the fact that the ratings of teacher opportunity to learn were high (more than 85%) in both countries.

(Item 076) Four 1-liter bowls of ice cream were set out at a party. After the party, 1 bowl was empty, 2 were half full, and 1 was three-quarters full. How many liters of ice cream had been EATEN?

A 3*3/4 B 2*3/4 C 2*1/2 D 1*3/4 E None of these

Responses	A	B	C	D	E*	TOTL
USA	6	30	9	23	30	85%
JPN	4	30	6	25	32	99%

The above problem is a typical word problem in everyday life. The p-values for this item are 30% for the U.S. and 31% for Japan. The patterns of responses are similar for both countries. On the other hand, when many of the Japanese students had chosen their answer from the five alternatives for the following item, they had some confusion or misunderstanding.

(008) In a school of 800 pupils, 300 are boys. The rate of the number of boys to the number of girls is

A 3:8 B 5:8 C 3:11 D 5:3 E 3:5

Responses	A	B	C	D	E*	TOTL
USA	25	9	6	10	46	91%
JPN	24	3	1	41	31	98%

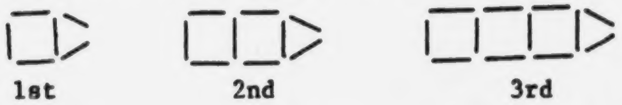
The p-values of this item are 46% in the U.S. and 31% in Japan. Achievement of the U.S. students for this item is greater than that of Japanese students. Many Japanese students took choice D instead of the correct answer E.

4.2 Achievement in Analysis Level

The following eight items were included in the analysis level which was defined to include the solution of non-routine problems, discovering relationships and formulating generalizations.

Analysis Items for Population A: (Code Number)

(002) Matchsticks are arranged as follows



If the pattern is continued, how many matchsticks are used in making the 10th figure?

A 30 B* 33 C 36 D 39 E 42 (* correct answer)

(018) If $6x - 3 = 15$

then $6x = 15 - 3$ (i)

and $6x = 12$ (ii)

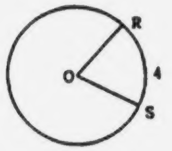
and $x = 12/6$ (iii)

and $x = 2$ (iv)

The error in the above reasoning, if one exists, FIRST APPEARS in line

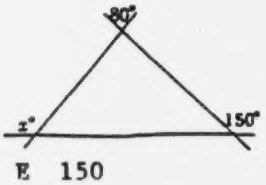
A* (i) B (ii) C (iii) D (iv) E None of these, there is no error

(023) The length of the circumference of the circle with center at O is 24 and the length of arc RS is 4. What is the measure in degrees of the central angle ROS?



A 24 B 30 C 45 D* 60 E 90

(059) Three straight lines intersect as shown in the diagram. What is x equal to?



A 30 B 50 C 60 D* 110 E 150

(074) 1st row 1

2nd row 1 - 1

3rd row 1 - 1 + 1

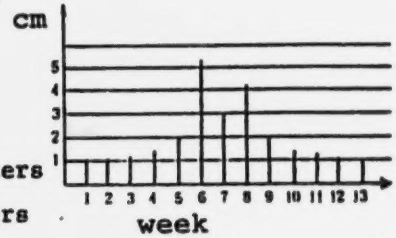
4th row 1 - 1 + 1 - 1

5th row 1 - 1 + 1 - 1 + 1

What is the sum of the 50th row?

A* 0 B 1 C 2 D 25 E 30

(099) In the graph, rainfall centimeters is plotted for 13 weeks. The average weekly rainfall during the period is approximately



A 1 centimeter B* 2 centimeters

C 3 centimeters D 4 centimeters

E 5 centimeters

(144) One bell rings every 8 minutes, a second bell rings every 12 o'clock. In how many minutes will they next ring together?

A 8 B 12 C 20 D* 24 E 96

(168) A solid plastic cube with edges 1 centimeter long weighs 1 gram. How much will a solid cube of the same plastic weigh if each edge is 2 centimeters long?

A* 8 grams B 4 grams C 3 grams D 2 grams E 1 gram

The responses for each item, with the rates of teacher opportunity to learn and teacher estimate of student achievement is given in Table 7.

Table 7 Responses of Analysis Items for Population A

Code		Responses of Student (%)					Teacher (%)	
		A	B	C	D	E	TOTL	TESA
002	USA	21	40*	11	6	18	34	31
	JPN	13	61*	12	2	8	44	34
018	USA	42*	13	14	10	17	74	36
	JPN	77*	3	7	2	9	100	69
023	USA	17	17	17	30*	11	24	19
	JPN	3	4	8	75*	6	93	49
059	USA	10	11	14	15*	46	43	23
	JPN	4	6	7	51*	25	53	42
074	USA	34*	36	6	17	5	21	25
	JPN	41*	34	5	12	5	20	27
099	USA	15	47*	16	11	9	73	41
	JPN	10	67*	11	3	6	81	46
144	USA	6	12	18	40*	21	49	28
	JPN	3	5	8	72*	10	97	47
168	USA	10*	20	7	59	2	58	30
	JPN	35*	26	2	36	1	75	43

Note: * : Correct responses
TOTL: Teacher "opportunity-to-learn" index
TESA: Teacher estimate of student achievement

Differences are calculated by subtracting the percentage of correct responses for each item from the corresponding teacher estimate of student achievement (TESA) and are almost all negative, excepting items 059 and 168 for the U.S. and 168 for Japan. This means that teachers under-estimated the performance of their students. Items 002, 023, 099 and 114 for Japan are especially negative with absolute values more than 20%; that is, student performances are higher than teacher estimates.

Items 002 and 074, involving the understanding of a sequence expressed in diagrammatic form, could be done by simple induction or drawing and counting rows and do not necessarily involve any mathematics which should have been specifically taught. Students would have been familiar with diagrams of this type, but teachers seemed to think that they had not taught such topics as sequences or series belonging to upper secondary mathematics. In fact, the ratings of teacher opportunity to learn are relatively low for both countries. In general, percentage of the correct answer at the analysis level are relatively lower than those of other levels for both countries.

6. Summary

As a brief summary of this paper, comparisons of achievement in connection with the curricula of both countries are illustrated and are of substantial interest regarding goals, teaching activities and teaching styles, given the current concern of the quality of mathematics education in both countries. It identified ways in which U.S. mathematics teachers differ from Japanese mathematics teachers in teaching practices. In addition, it identified the differences in students' achievement by items, and subscores for both countries. It seems to me that the differences between the attained and implemented curricula will have both positive and negative implications.

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Discussion of Mr. Sawada's Paper

Wilson: We are actually about three minutes early, so Mr. Sawada has made very efficient use of his time. I would personally like to compliment Mr. Sawada on his very clear presentation and, speaking for the rest of the U.S. delegation, I can say we were following the talk very easily. I would remind each of you, as you have a question, turn on the microphone, then state your name, ask the question and then give the translator an opportunity to translate. Who would like to go first?

Silver: I'd like to pick up on the comment that you made at the very end of your talk. In your paper you mentioned that the IEA results have both positive and negative implications and I was wondering if you would discuss some of those implications briefly for us.

Sawada: On the negative side there is a difference between teachers' understanding of what the students actually learned - intended and actual. We must clear up the cause of the teachers' underestimate or overestimate for students' learning. For example, in item 008, there is a difference between the teacher's estimate and students' actual achievement.

Hashimoto: I'd like to give an example - how about item 008 on page 133? In this problem the achievement of U.S. students is higher than that of Japan. Why do 41% of Japanese students pick the wrong answer, choice d? For United States students, only 10% picked this response. I think this is one example of an implication from the items. Such a problem is taught at the fifth grade in Japan. Classroom teachers do not emphasize topics on proportions like this. This is one implication.

Wilson: I would point out that the word is ratio.

Rachlin: I just wanted to check on that same item. I'm not convinced that the U.S. students who got it correct did so because they know about ratios. It may be that the way the question is stated makes it easy to pick 3:5 as the correct ratio because 300 is the given number and they are looking for some other number, so they place it next. I am wondering whether, after translation, it may be a harder problem in Japanese than it is in English?

Hashimoto: I think one of the reasons is correct. Mr. Sawada, what do you think?

Sawada: It may be that Professor Sugiyama will give comments on such content based on Japanese textbooks this afternoon. In Japan, ratio and proportion are covered in the elementary school, but the students' achievement is generally lower than the teacher's estimate.

Rachlin: I am wondering whether the differences in the comparative results derive from the wording of the item. What I was trying to get a feel for is whether the differences result from the translation into Japanese - perhaps, there is a different style of presentation?

Sawada: In a school of 800 people, 300 are girls. The ratio of number of boys to the number of girls is more correct. This sort of comparison between the number of girls and the ratio of boys to girls is not found very commonly in Japanese mathematics' problems, for example. So the presentation sounds strange.

Nohda: In the U.S. the custom is to say ladies and gentlemen, whereas, in Japan, we say gentlemen and ladies. This may also play a role here.

Wilson: I'd like to comment. In general, ratio and proportion is not covered very much in the school curriculum in the United States. The research literature on students learning ratio and proportion concepts shows that one of the difficulties is that kids continue to use an additive strategy. If you apply that wrong additive strategy to the way this problem is worded, you'd still get the right answer. I don't believe it reflects very much what students know about ratio, and it may be that they do better because the wrong strategy also gets the correct answer. That's one very biased opinion based on my perception of the item.

Silver: One other comment about this item. The most common error that students make in ratio problems of this sort is the reversal error; that is, to say 3:5 instead of 5:3, or vice versa. If you look at the performance the Japanese students, actually 72% of the Japanese students are using the right numbers, and we wonder why the American students are choosing different numbers in their answers. I think the reasonable choices are d and e and the Japanese students are choosing d or e more than the American students.

Inouye: I am surprised that so many American students got it right, based on my experience. They usually get it wrong. They have to do the subtraction of 800 - 300. They cannot take that 300 from 800.

Becker: Usually by about the middle or junior high school years in the U.S., students are learning to translate from verbal statements of problems into mathematical notation in a kind of one-to-one correspondence manner from words to symbols. The students are learning a certain way of mapping words onto mathematical symbols. Now I can't say that it is actually happening at the fifth grade level in the U.S., but there is a certain flow of the words to the symbols here. The problem states that in a school of 800 pupils, 300 are boys, the ratio of the number of boys (300) to the number of girls is to be filled in. If we look at the responses that follow that pattern, the plausible ones are a and c and e, and e is correct and that gives you a total of about 77%, I think. So I would expect American students to respond that way.

Travers: I'd like to make a general observation about the issue. One of the many fascinating issues about the IEA study is how one deals with the linguistic and cultural issues and barriers, if one would like to regard them in that way, in testing achievement across countries. As part of the IEA standard procedure, after the item pools have been developed and pilot tested, there are fairly elaborate procedures for translating the items into the native language and independently back translating the items. I'm convinced that one of the many important kinds of analyses that should be done is to look at some of these issues in terms of how the translations have been handled and possible linkages, in this case, between the language structure and achievement. I think this is just one example of an important kind of subanalysis that somebody really should be doing.

Whitman: I guess this is directed toward Mr. Sawada and Professor Travers. I am curious whether there were any data collected on when the topics are covered in the classroom relative to the time that testing took place. I notice that Mr. Sawada says that this is generally covered in the fifth grade in Japan. When is it covered in the American population?

Travers: Yes, we have data on when the topic was covered with respect to the school year because we asked the teachers "Did you teach this content during the year and if not, why not?" One of the options was that it not be prior to the school year. The students were also asked whether they had seen this material this year or in the prior year. So that there is something to that.

Inouye: Do they actually make use of the data?

Travers: Use is made of the teacher data, teacher opportunity to learn data; but to my knowledge, student opportunity to learn data has been used only to look at the extent to which the ratings of teachers and the students correspond. But it hasn't been broken down into looking at sequencing issues.

Miwa: Please look at page 136 where I find some interesting data in Table 7. For example, for item 018 the U.S. distribution of responses is rather uniform except for the correct response; but, for items 002 and 144 the distributions are not uniform but concentrate on two choices. And the percentages for correct answers are nearly equal. What do you think about the differences?

Sawada: Yes, it is true that the percentage of OTL (opportunity to learn) and TESA (teacher's estimate of student achievement) are very similar and low in both countries. I think that teachers have not taught the topics, but students are doing them by simple induction, such as in item 002. On the other hand, it seems to me that item 144 depends on OTL and such a topic has been taught in classrooms in Japan. As a result, I think high achievement on item 144 depends on high OTL in Japan.

Miwa: Perhaps American members might make some comments about these differences.

Wilson: Shall I just call on one of the American members to comment?

Becker: Item 002 involves patterns and induction and the Japanese scored considerably higher than the U.S. students. That doesn't surprise me because I believe that Japanese primary school students receive more experience with that kind of learning than their American counterparts. In fact, if I am to generalize from some of the material I have read about Japanese primary education and what various Japanese mathematics educators have said to me, there is quite a lot of emphasis on inductive reasoning in the primary school curriculum in Japan.

Kantowski: Since we are just guessing here, the original observation was that the wrong answers were pretty evenly distributed in 002 and the incorrect answers were clustered in 1 or 4. In 002, as Dr. Becker indicated, the students have difficulty with pattern recognition. There is no way that one could guess what to do with the numbers. Regarding 144, students in the United States are notorious for taking numbers and just doing something with them. One thing that you can do in 144 is to add the two numbers 8 and 12 and get 20. That was one of the high percentage responses. You could also multiply those numbers and get 96 and that was also another high percentage response. So I think that one of the reasons for the cluster was simply that the students did something with the numbers if they couldn't solve the problem.

Wilson: I don't see the cluster. Those look like exactly the same distributions - one being 11 and the other one 12. But if you look at the four incorrect responses, you've got the same numbers across them. I don't see where one is more clustered than the other. Am I missing something?

Becker: Yes, I think you are.

Wilson: What am I missing?

Becker: If I understood what Mary Grace is saying correctly, she is giving an explanation of why the choices would be made for various distractors and she is arguing that typically American kids will just start to do some things with numbers, just to operate on them. And there is clear evidence in the response pattern, that students may be doing that.

Wilson: The previous observation was that they were more clustered in item 144 than in item 002, I wonder about that. I don't think they are. Now I don't disagree with the analysis of 144 about taking numbers out of context in the problem and putting them together to get erroneous responses. That's how those distractors are written in the first place. If you look at 002, you could argue that in fact the two responses that are picked the most, a and e both reflect some degree of inductive reasoning on the part of those students. They just made an error in some assumption in the process. One is 3 x 10, you get the 10 figure you are looking for and you add 3 each time and that's the 30. It jumps out at you. Regarding the other one, there's four on each box, and they've got two that form the point so that's 42. There is a pattern that could be guessed by inductive reasoning. I don't know anybody that sat down with some students and let the students talk through to see what they come up with. I don't disagree with what Mary Grace was saying about her analysis on that problem, but I think the previous observation that they were more clustered in one item than the other may not be on the mark.

Silver: I would like to make a couple of comments. I think that as I looked across these items, being spurred on by Professor Miwa's original observation, it seems that this might be an interesting basis for a study to examine differences between the U.S. and Japanese populations in terms of their susceptibility to certain distractors or their tendency to choose certain distractors. In most of these items it seems that, except for the fact that the Japanese students choose the correct answer much more frequently than the U.S. students, there is a fair amount of consistency across the two populations in choosing the wrong answer choices. An exception is item 168 where the selection of choice b is higher for the Japanese population than it is for the U.S. population. This might be a basis for some interesting analyses that could be done. Repeating the statement, if you look at the items and focus on the wrong answer choices and compare the U.S. and Japanese students, in general they perform in a similar way with respect to choosing wrong answers. In other words, the ratio of choices of wrong choices from the Japanese population to the U.S. population is relatively constant, with maybe twice as many American students making the wrong choices as Japanese, but that ratio remains constant across all the wrong choices. Except in the case of item 168, it's different. I wonder if anyone has some speculation about why choices b and d have a different frequency of selection in the U.S. and Japanese populations?

Rachlin: One of the things that bothers me in Dr. Silver's analysis is that the biggest difference is that this is an item in which there is a large frequency of incorrect responses for the Japanese group. Because of the larger number of incorrect responses, it is going to change the ratios that you see later on. But in terms of the relative errors that are occurring, if you just look at the errors, d is the most popular error that is made and b is the next most popular error and that's consistent between the two. That might be more important in what are the rankings of errors as opposed to actual ratios. In looking at 168, I don't think that it is relevant to look at a ratio of U.S. to Japanese score because the major thing that's changed is that there are more errors for everybody in that one. But what I liked in what was being suggested is looking at how the errors are clustered for the two groups to see if the U.S. most popular error is the same as the Japanese most popular error. It would be an interesting thing to look at.

Becker: I have a comment about #144 again. When I read that problem it just strikes me as a thinking kind of problem. Now when I look at the data given on page 136 then it is clear that the Japanese students choose the correct answer almost twice as often as American students and it looks like they apply the wrong strategy far less frequently than American students. That is, it looks like they are not just adding the two numbers or multiplying the two numbers as often as the American students. And that doesn't surprise me at all.

Hashimoto: I understand what Professor Becker said. From all the results, what kind of things can we deduce about classroom teaching? If so, it's no surprise that it is not so much creative, but what kind of constructive view can we deduce from these results? Are there other comments from the American delegates?

Becker: One of the things that I infer from this, and I infer it also from various parts of the literature in mathematics education and from what knowledgeable Americans say about Japanese mathematics education, is that we need in the United States to place far more emphasis on developing thinking skills from the time kids first enter primary school. We have seen in Professor Hashimoto's talk, in the writings of Jack Easley, and other sources that Japanese primary teachers seem to spend more time on thinking type exercises in classes than I think we do in the United States. I want to be careful and not overgeneralize, but that's an impression I have.

Travis: Also I think that one of the implications for classroom teachers in the U.S. is to impress upon them the importance of error analysis, because one of the advantages of a multiple choice item is that you can learn so much from the errors that students are making. Perhaps there is something in the teaching interaction that the teacher can do differently once he or she can analyze the errors that are being made - to learn from those and then change something within our style of teaching. Perhaps the student is doing problem solving with some kind of emphasis on pattern finding, induction and so forth, but perhaps it is given in such a way that students are not picking up the correct aspects. So I think looking at the errors can be very insightful for teachers.

Wilson: I want to respond to that. I think if you look at these two items, 144 and 002, in the sense of analyzing errors as is being suggested, my judgment about 144 is that the analysis of those errors means that students didn't think about the problem. The distractors are designed so that you aren't thinking about it. Students have been drilled on multiple choice tests in the U.S. curriculum like this. These look alot like textbooks from which kids have learned all kinds of abortive strategies to respond to numbers. I think all we get from the error analysis on that item is that they made silly mistakes of the obvious kind, combining two numbers in some way. On the other hand, an analysis of error patterns for 002 show that they are partially correct strategies that somehow lead to those distractors. My point is that I don't think we learn a whole lot from the error analysis of 144 where we might have the opportunity to learn more, or at least have ideas to test, and talk with students on 002.

Kantowski: I would like to comment about two of the other items. In 059 and 168 the p-values are less than random in the case of the U.S. students and it is important to note that those items were two of the three geometry items in that set. What that is saying is that we did a really poor job and that we need to look at our instruction in geometry. There were only 3 geometry items in this whole section and those are 2 of the 3.

Wilson: I want to point out that the distractor choices are not random.

Silver: This is a point of information. Would those items be coded as geometry items or as measurement items? One involves angle measurement and the other involves either weight or volume, depending on how you look at it.

Sawada: Items 023 and 059 are geometry items. The ratings of TOTL are low. Japanese teachers say they have not taught these items at this stage. But the percentage of Japanese students saying they had an opportunity to learn are higher than those of teachers. Students thought that they had already learned such problems in elementary school.

Wilson: Professor Shimada, you had your hand up once and I didn't come back to call on you - do you have a question or comment?

Shimada: I have two comments. One is on the translation of test items, the other on the classification of items in the behavioral dimension.

While we tried to translate the original items in English into the languages of participating countries preparing the IEA study, we found it very difficult to decide at what level of language formality they should be translated. Perhaps the situation might be the same in other non-English speaking nations. As you know, there are many levels of formality in a language. Home or peer language may be the most informal one. The second one may be classroom language which seems a little more formal than home language. The third or other higher levels may be used according to the needs of formality in such situations as public speaking, writing, academic presentation, legal documents in business, etc.

The level of formality of language used in the testing is usually a little more formal than that of the classroom language in Japan, though in that case there may still be some difference in level. All Japanese newspapers report problems that are set in the entrance examinations, and if there are ill-formulated ones or ones with ambiguous expressions, they are severely criticized. So in formulating test items which are to be published afterwards, we must be careful about the use of language. In translating the IEA test items, we adopted the usual level of formality used in other tests published widely. The language difficulty in the item no. 008 may be partially due to this policy.

For the classification of items in the behavioral dimension, I recall the discussion in the IEA Council Meeting for the IEA six subjects study in 1967. In that meeting, I questioned if the classification of test items in the behavioral dimension was free from what students had learned up to the time of testing. The answer was that it did not depend on what was learned, but only on the structure of the items. The structure of the item determines its category in classification. But I could not and cannot agree with this. For example, the item 144 is classified as 'analysis,' the highest cognitive level in the report. Surely if a student has no experience of solving this type of problem, then his mental behavior may be said to be 'analysis.' But for Japanese students who have learned the L.C.M. in grade 7 and had exercises like this, this would be a familiar type though the situation is not so popular in Japan. Therefore, if I am asked to classify it independently for Japanese students, I will classify it as 'application.' The classification which is internationally valid is very difficult, and the result is a kind of compromise.

Wilson: I think we will have to end this session now. Thank you.

Professor Traver's Paper

Hashimoto: Now we will begin the next session. I would like to introduce Dr. Travers to you. He has been a Professor at the University of Illinois at Urbana-Champaign since 1971. He has the experience of teaching in both the elementary and secondary schools and he took his degree from the University of Illinois in 1965. He is very famous through the work of IEA studies, as you know. He is a very active person and has travelled over the world. He has visited Japan two times, first in 1978 as the Chairman of IEA mathematics study and second in 1983 as the plenary session speaker of ICMI-JSME regional conference. He has written many books and recently published a book whose title is Mathematics Teaching. This is the second version and an excellent book for classroom teachers and students.

Travers: Thank you very much, Mr. Hashimoto, for that kind introduction. I would like to take this opportunity to put into the record, as it were, the deep appreciation of the IEA and the International Mathematics Committee for the very important work that Professor Shimada has made to both the first international mathematics study and the second study.

THE TEACHING AND LEARNING OF MATHEMATICS IN JAPAN AND IN THE UNITED STATES: SELECTED FINDINGS FROM THE SECOND INTERNATIONAL MATHEMATICS STUDY

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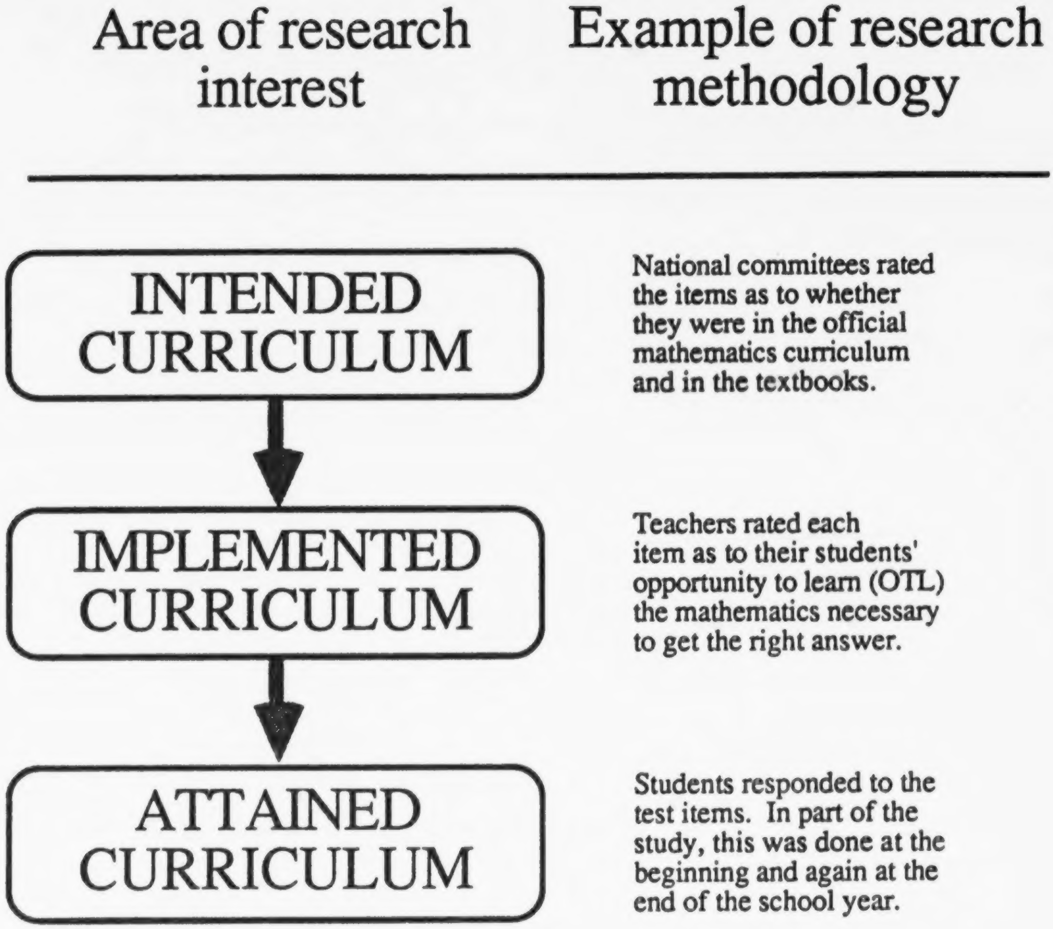
1. Introduction

The Second International Mathematics Study (SIMS) provides a rich source of data for examining various aspects of the teaching and learning of mathematics. In this paper, selected findings are discussed in the light of the focus of this Seminar on problem solving. Details are provided in Travers (1986) as well as in the national reports for Japan and the United States (see References).

The three-tiered view of the curriculum that provides the basis for SIMS is a useful rubric for the present paper. Each of the three levels of the curriculum (intended, implemented and attained) is a data source that warrants attention (see Figure 1). To each of these levels may be associated a variety of contextual factors that impinge upon schooling in general and mathematics education in particular.


Area of research interest
Example of research methodology
INTENDED CURRICULUM
IMPLEMENTED CURRICULUM
ATTAINED CURRICULUM

National committees rated the items as to whether they were in the official mathematics curriculum and in the textbooks.

Teachers rated each item as to their students' opportunity to learn (OTL) the mathematics necessary to get the right answer.

Students responded to the test items. In part of the study, this was done at the beginning and again at the end of the school year.

Figure 1 Overview of the IEA Second International Study of Mathematics

The intended curriculum is reflected in curriculum guides, course outlines, syllabi, and textbooks adopted by school systems. In most countries, national curricula emanate from a ministry of education or similar body. In the U.S., such statements of intended goals and curricular specifications come from state departments of education and from local districts. Thus it was considerably more difficult to describe the intended curriculum for the U.S. than for almost any other country that took part in the Study.

The implemented curriculum focuses on the classroom, where the teacher interprets and puts into practice the intended curriculum. Teachers exercise their own judgment in translating curriculum guides and adopted textbooks into programs for their classes. Hence, their selection of topics or patterns of emphasis may not be consistent with those intended.

To identify the implemented curriculum, a number of questionnaires were developed for classroom teachers to complete. For example, teachers were asked whether or not they had provided instruction for each of the items on the achievement tests. They were questioned about such matters as the use of calculators in their classes. They were also asked to provide detailed information on the number of class periods that they devoted to specific topics and subtopics and on how they presented and interpreted this mathematical content to their classes.

The attained curriculum - what students have learned as measured by tests and questionnaires - was the third component of the Study. Extensive achievement tests were designed to assess student knowledge and skills in areas of mathematics that were designated as important and appropriate for the students being tested. The "fit" between these tests and the actual curricula in individual countries varied considerably, because the tests contained items that were less appropriate in some countries than in others and because they could not possibly contain an adequate range of items to fully represent all curricula in all countries.

The student outcome measures also included a number of opinionnaires and attitude scales. These were devised to elicit students' views on the nature, importance, ease, and appeal of mathematics in general and of selected mathematical processes.

Questionnaires on background information were designed for schools, teachers and students, as indicated in Figures 2 and 3 for the two target populations of students, Population B and A, respectively. Item sampling was utilized in order to provide for sufficient content coverage.

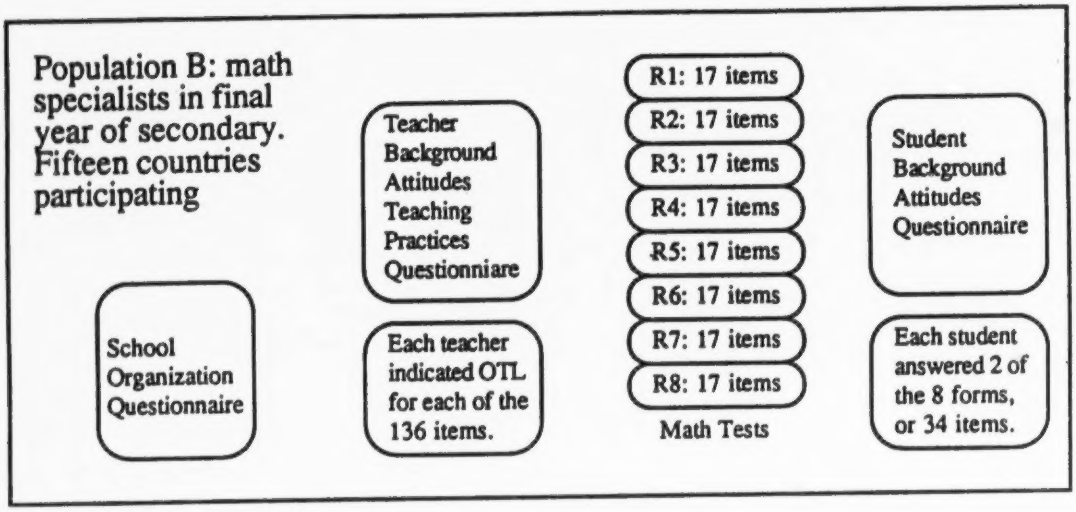

Population B: math specialists in final year of secondary. Fifteen countries participating
School Organization Questionnaire
Teacher Background Attitudes Teaching Practices Questionniare
Each teacher indicated OTL for each of the 136 items.
R1: 17 items
R2: 17 items
R3: 17 items
R4: 17 items
R5: 17 items
R6: 17 items
R7: 17 items
R8: 17 items
Math Tests
Student Background Attitudes Questionnaire
Each student answered 2 of the 8 forms, or 34 items.

Figure 2 Survey Instrumentation for Cross-Sectional Study, Population B

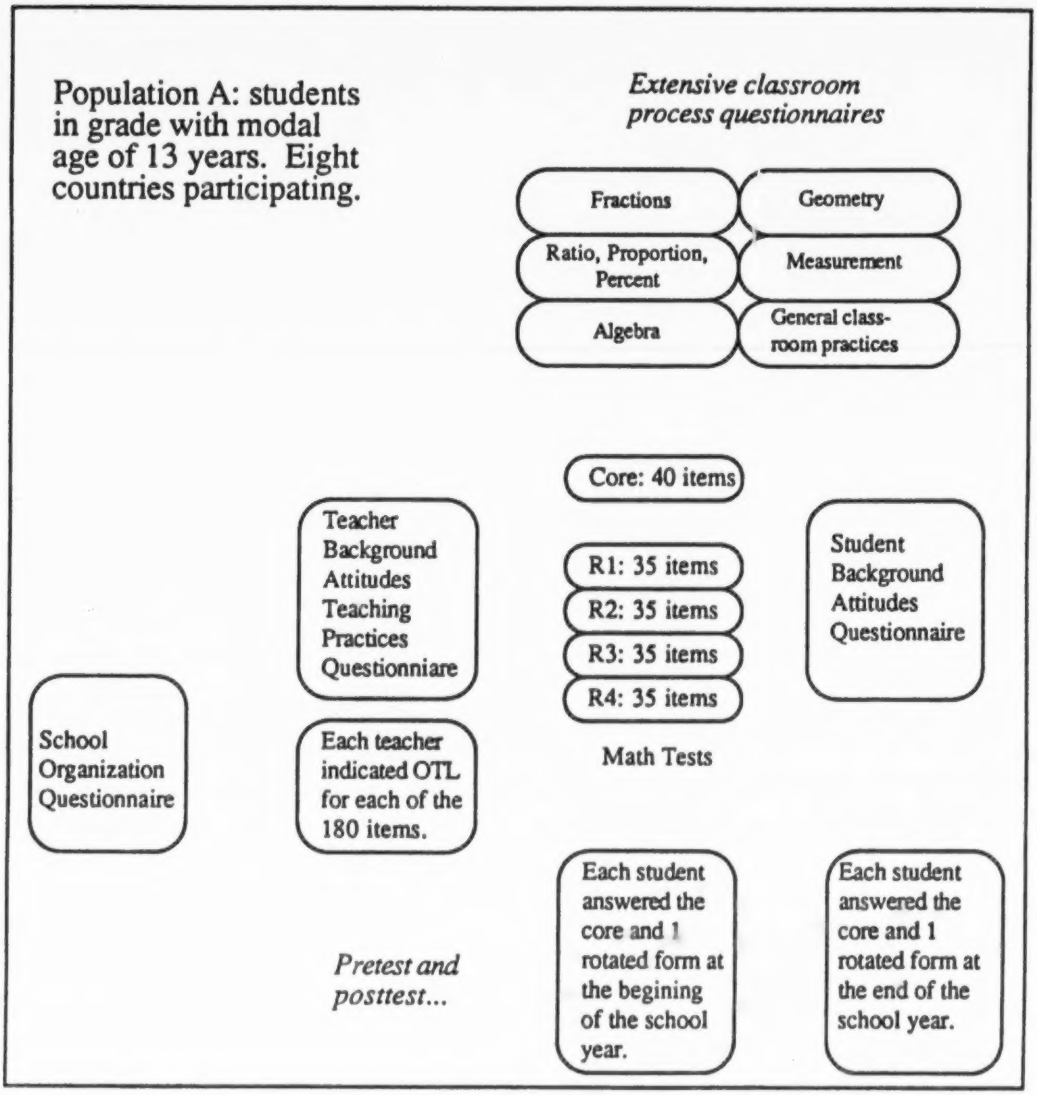

Population A: students in grade with modal age of 13 years. Eight countries participating.
Extensive classroom process questionnaires
Fractions
Geometry
Ratio, Proportion, Percent
Measurement
Algebra
General classroom practices
Core: 40 items
R1: 35 items
R2: 35 items
R3: 35 items
R4: 35 items
Math Tests
Teacher Background Attitudes Teaching Practices Questionniare
Student Background Attitudes Questionnaire
School Organization Questionnaire
Each teacher indicated OTL for each of the 180 items.
Pretest and posttest...
Each student answered the core and 1 rotated form at the begining of the school year.
Each student answered the core and 1 rotated form at the end of the school year.

Figure 3 Survey Instrumentation for the Longitudinal, Classroom Process Study in Population A

In this paper only a few of the contextual factors are selected for discussion. Most of them are chosen because they are of interest to many mathematics educators in the United States. Hopefully, they are of some interest to our Japanese colleagues, as well.

2. The Context of Mathematics Teaching and Learning

2.1 System Characteristics

2.1.1 Retentivity

IEA studies have consistently shown system retentivity (proportion of age cohort in school) to be a powerful factor in accounting for between system differences in mathematics achievement. That is, less retentivity (more selectivity) is accompanied by higher achievement. In this light, it is important to note that Japan retains 92% of seventeen-year-olds in school while the corresponding figure for the U.S. is 82% (see Table 1). The retentivity figure for Japan, the highest of any SIMS country, is particularly noteworthy in that it has increased from 50% in the mid-1960s (Fetters, et al., 1983). During the same time frame, U.S. retentivity at the system level increased only about 12 percentage points, from 70% in 1964, the date of the First International Mathematics Study (FIMS).

Table 1

Proportion of Students in Relevant Age Groups and Grade for Each Country: 1981 (Population B)

Country	Age Group (Years)	Pop B Percent of Age Group	Pop B Percent of Grade Group	Percent of Age Group in School
Belgium (FL)	17	9-10	25-30	65
British Columbia	17	30	38	82
England & Wales	17	6	35	17
Finland	18	15	38	59
Hungary	17	50	100	50
Israel	17	6	10	60
Japan	17	12	13	92
New Zealand	17	11	67	17
Ontario	18	19	55	33
Scotland	16	18	42	43
Sweden	18	12	50	24
U.S.A.	17	13	15	82

Notes: 1. Age group is estimated age at middle of school year.

2. While the fourth column represents the percent of the age cohort still in school, this does not imply that all these students are in the grade(s) from which the Population B sample is drawn. Thus the second column is not always a simple product of the third and fourth columns.

3. Data are obtained from national reports for each country. The ratio of high school graduates to population age 17 was 72 percent in the United States in 1981. U.S. data on enrollment were based on the school enrollment rates of persons 17 years old according to the October 1981 Current Population Survey. An additional 5 percent was enrolled in college or university.

2.1.2 Target Populations

Two target populations were defined for the Second International Mathematics Study:

> Population A: All students in the grade level where the modal number has attained the age of thirteen years by the middle of the school year.
>
> Population B: All students who are in the normally accepted terminal grade of the secondary education system and who are studying mathematics as a substantial part (approximately 5 hours per week) of their academic program.

In both countries, Population A encompassed virtually all children. It is noted, however, that, in Japan, Population A consisted of twelve year olds instead of thirteen year olds (as it was in the U.S. and almost all other countries). Two main reasons are given for choosing a younger age group (seventh grade) in Japan. In the First International Mathematics Study (1964), testing occurred in May. Since Japan's school year begins in April, their testing was at the beginning of the eighth grade (grade in which most thirteen-year olds were enrolled). Furthermore, since in the Second Study, testing was done according to each country's school year, end of year testing for seventh grade in Japan corresponded most closely with the time of testing in the First Study. It was also found that the content covered by the international test provided a good fit to the seventh grade mathematics curriculum in Japan.

A note is in order for Population B. Even though the retentivity figures for Population B mathematics are similar in both countries, it should be pointed out that in Japan, all Population B students study calculus. In the United States, it is estimated that only about 20% of the Population B students are engaged in a full-fledged calculus (AP) program.

2.1.3 School year (length of hours of mathematics instruction)

The six day school week in Japan is reflected in their long school year. But since, at Population A, mathematics is studied, on average, about 30 minutes per day in Japan as compared with about 40 minutes per day in the U.S., the yearly amount of mathematics instruction is greater for the United States. The time spent in Japan in out-of-school tutoring is not reflected in these data (see Figures 4 and 5).

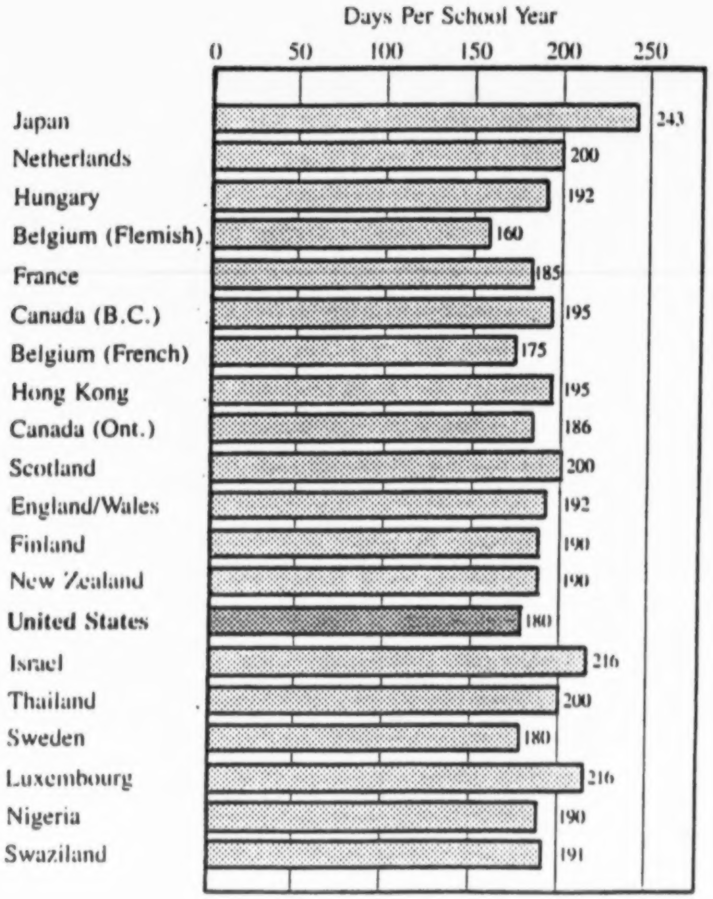


Figure 4 Median Number of Days per School Year (Population A)

Countries are ranked according to overall achievement on the international mathematics test.

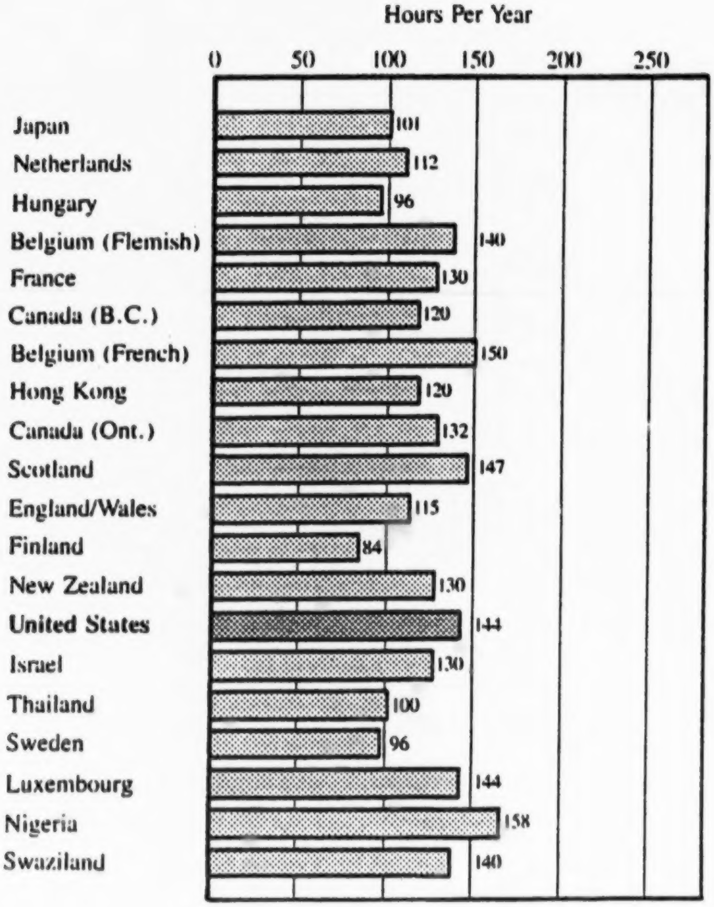


Figure 5 Hours of mathematics instruction per year (Population A)

2.1.4 Class size and gender

Mathematics classes are large in Japan (as well as in Hong Kong and Thailand) with average size of about 40 students (see Figure 6).

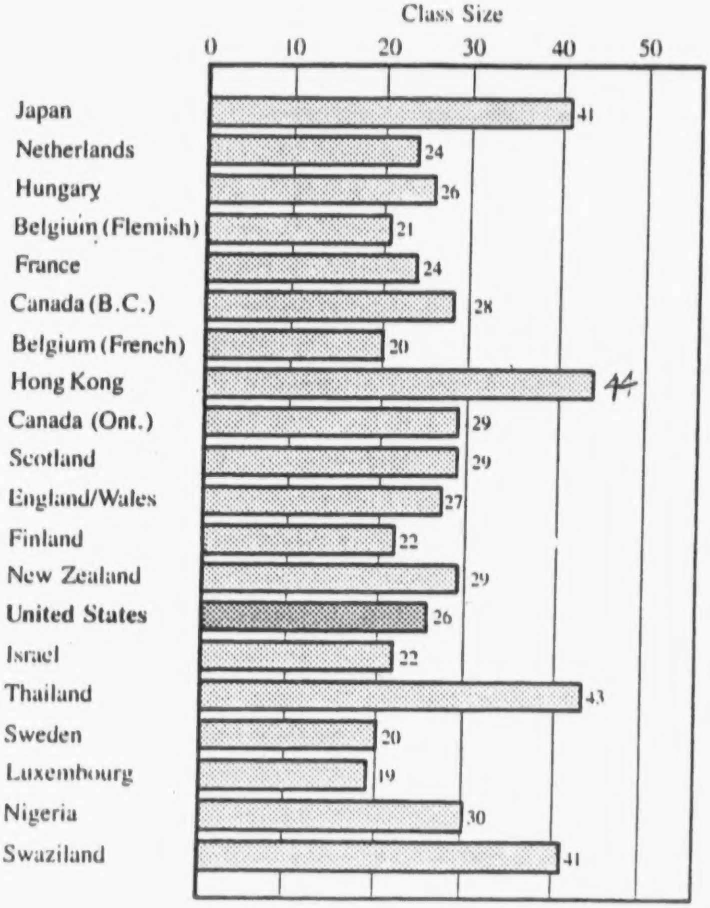


Figure 6 Median sizes of mathematics classes (Population A)

In Population A, the proportion of females in both countries is (as expected, due to the nature of the target populations) close to 50% (48.5% in Japan and 51.9% in the United States). However, in Population B, the proportion of females is 22% for Japan and 44% for the United States.

2.1.5 Homework

The amount of homework reported by students does not differ greatly between the two countries for either Population (see Figure 7).

Population A Students' Reports of Mathematics Homework Assigned in a Typical Week

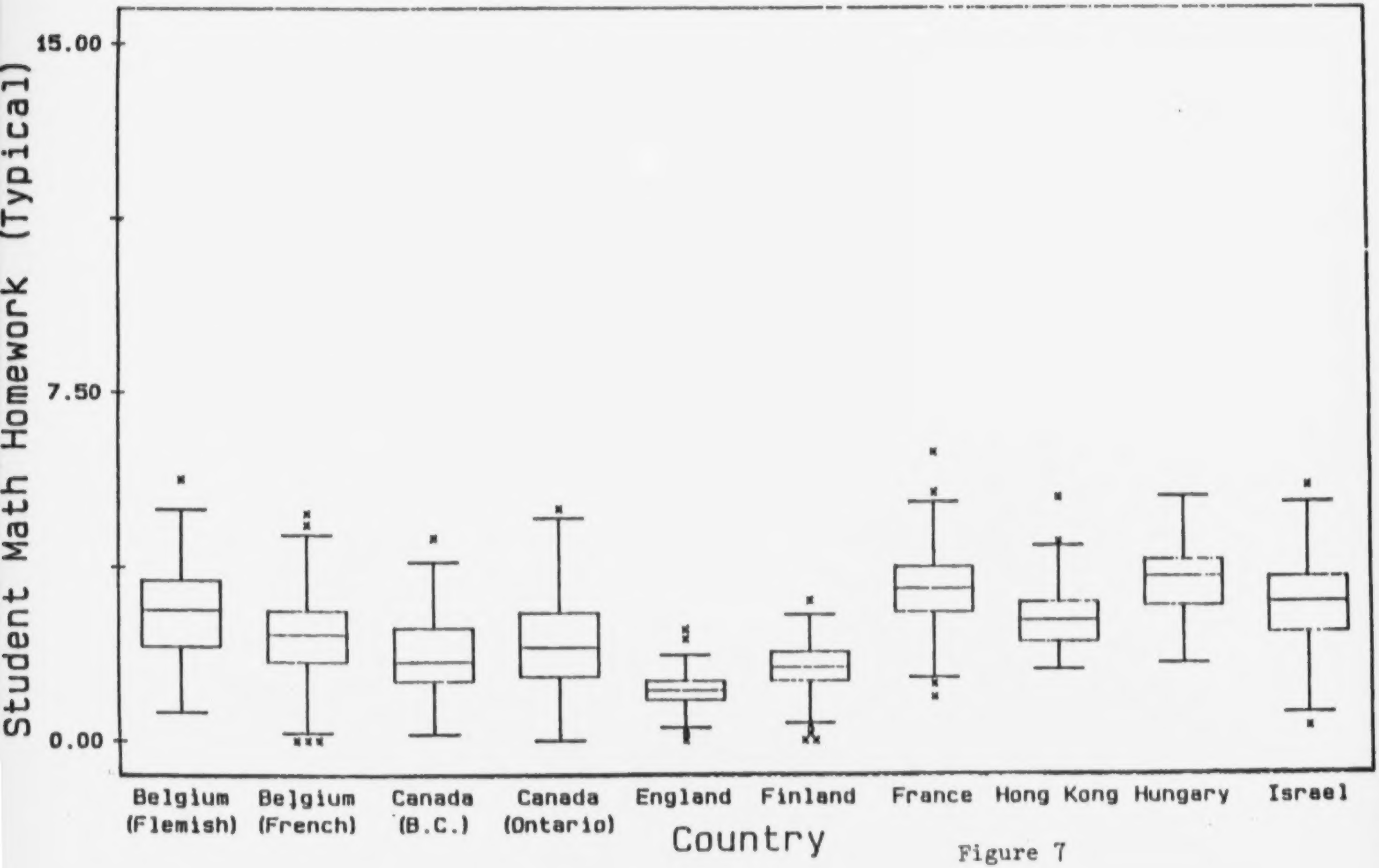


Figure 7

Population A Students' Reports of Mathematics Homework Assigned in a Typical Week

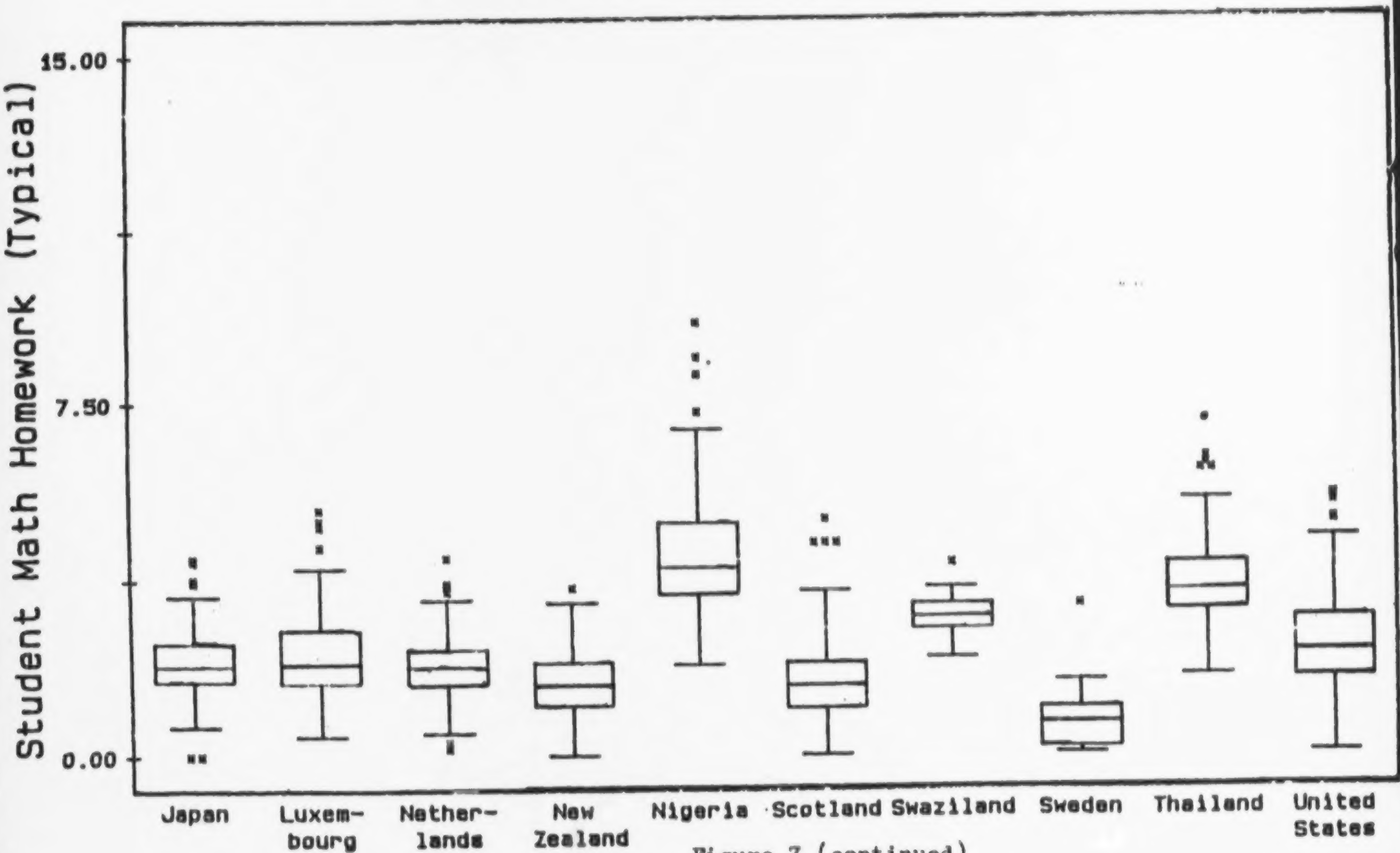


Figure 7 (continued)

2.2 Teacher characteristics

Extensive data on the teachers of the sampled classes were obtained from the teacher background questionnaires. These data included information on the "workplace" of the school and classroom, as well. Japanese and U.S. teachers are similar in terms of age and years of experience. However, there is some evidence that the professional preparation of Japanese mathematics teachers may entail a greater emphasis on pedagogy than that of the U.S. teachers (see Table 2).

Table 2

Teacher background data for U.S. and the comparative data for Japan.

Median Number of Semesters of Post-Secondary Mathematics Studied

	U.S.	Japan
Population A	9.3	6.0
Population B	16.4	8.0

Median Number of Semesters in Mathematics Methods and Pedagogy

	U.S.	Japan
Population A	2.0	4.0
Population B	2.4	4.0

Median Number of Semesters in General Methods and Pedagogy

	U.S.	Japan
Population A	3.8	4.0
Population B	3.8	4.0

3. The Content of the Intended Curriculum

On the surface, the content of the intended curriculum for each target Population A does not differ greatly between the U.S. and Japan. The significant differences in the content (within the limitations of the IEA approach) appear to be in degree of emphasis rather than in kind of mathematics offered. With respect to algebra, for example, similar topics are covered, in both countries, with two important contrasts: (i) it is offered one year earlier in Japan and (ii) it is presented with a great deal more "intensity" in Japan (see Figure 8).

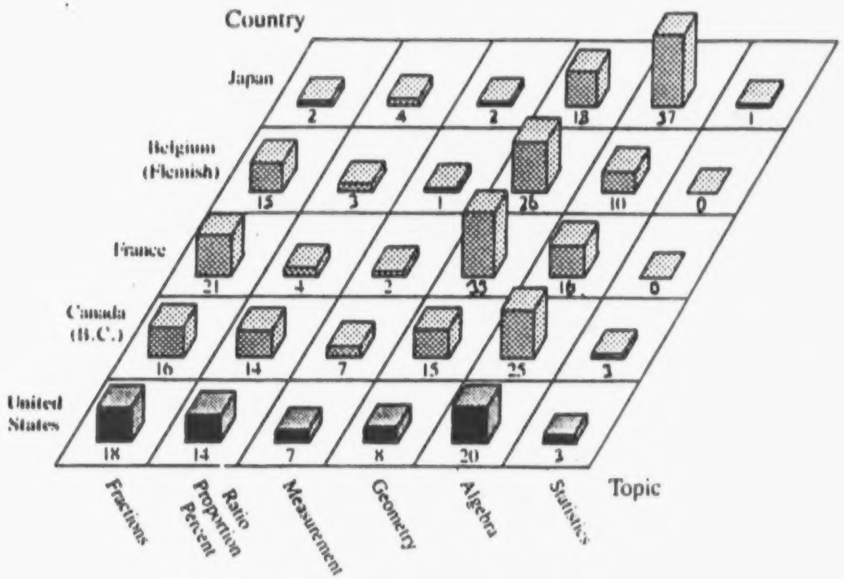


Figure 8 Intensity of mathematics instruction in Five Population A systems

The data in Table 3 are based on teacher reports of the number of periods devoted to geometry. Median polish has been used to help in identifying row (topic) and column (country) effects. For example, the relatively large negative effect for vectors indicates little attention paid to this topic except for France and Belgium. The large positive effect for spatial relations and solids in Japan indicates a special emphasis on these topics.

Table 3

Geometry Topics Taught or Reviewed

(Median Polish)

	BFL	CBC	CON	FRA	JPN	NZE	THA	USA	COLUMN EFFECTS
Angles	-16			-60	24				31
Transforms.	61	-41		39		41	-21	-36	
Vectors	112	-30		66				-21	-29
Pyth. Thm.		33		-51	-21		33	17	
Triangles	-30				-17				30
Polygons	-15						-20		20
Circles	-28						-26		21
Congruence				-20	-20		28		
Similarity				-51	-26		34		
Parallel Lines	40								33
Spatial Relations					72				-19
Solids					54		24		
Construction				17	32	-19	-16	-20	23
Proof	102	-26		64			51	-21	-31
Coordinates				15		26			
Row Effects	-50	-	-	-25	-	-	-	-	60

4. The Implemented Curriculum -- Opportunity to Learn Mathematics

For both countries, the character of the intended curriculum is reflected, not surprisingly, in the curriculum as is reported to be taught. In Japan, the median proportion of the IEA algebra items that were taught is high--the highest of all the countries. Furthermore, the range is low, indicating a high degree of equality in Japan in opportunity to learn algebra. In the U.S., the lower median level of algebra coverage and the greater range in coverage reflects, among other factors, the curricular differentiation (tracking) in eighth grade mathematics (see Figures 9 and 10).

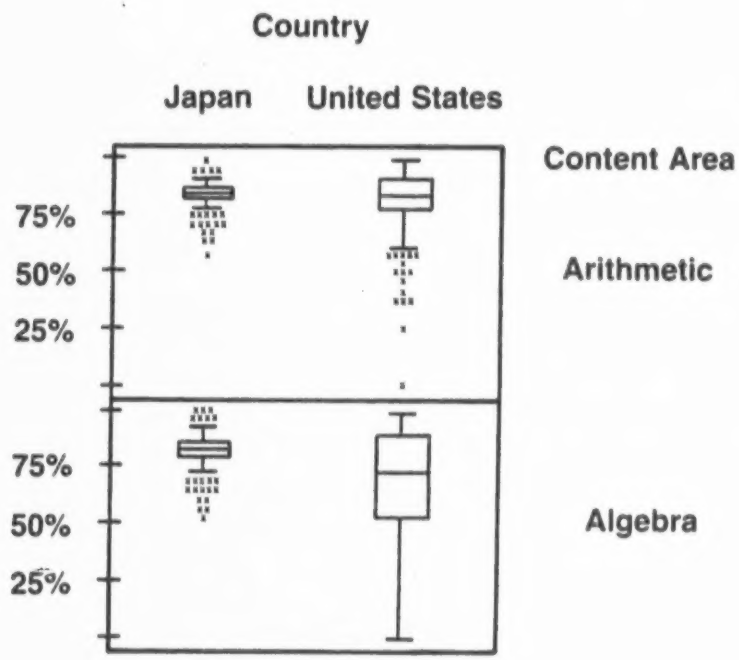


Figure 9 Opportunity to Learn Arithmetic and Algebra in the United States (Eighth Grade) and Japan (Seventh Grade)

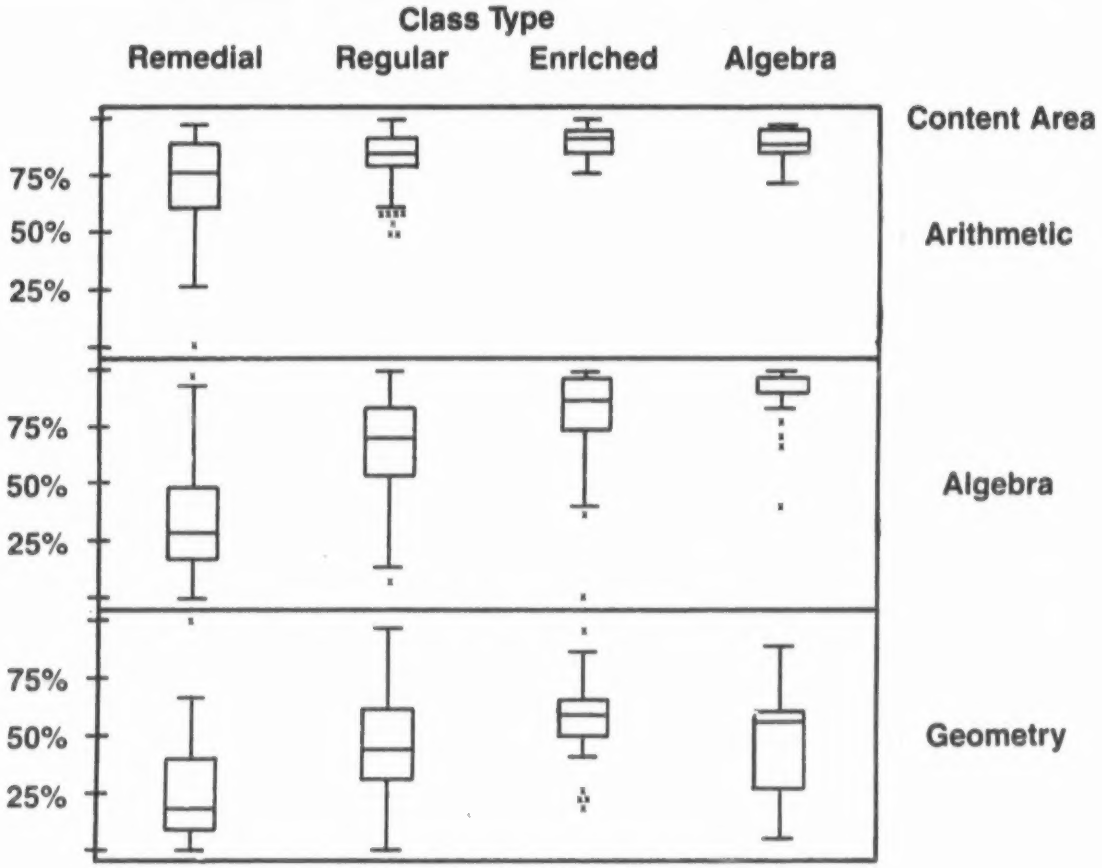


Figure 10 **Opportunity to Learn Mathematics: Class Type by Content Area (Eighth Grade, United States)**

A full analysis of the classroom process data at the international level has yet to be carried out. However, preliminary evidence indicates a predominance of symbolic-oriented procedures in U.S. classrooms. In Japan, by contrast, there appears to be more use of perceptual (concrete, materials-based) strategies.

5. The Attained Curriculum

Japanese students obtained the highest achievement scores, on average, of any IEA country for Population A (and was second only to Hong Kong in Population B). U.S. achievement was at or below the median in Population A (and typically among the lower quartile of countries for Population B).

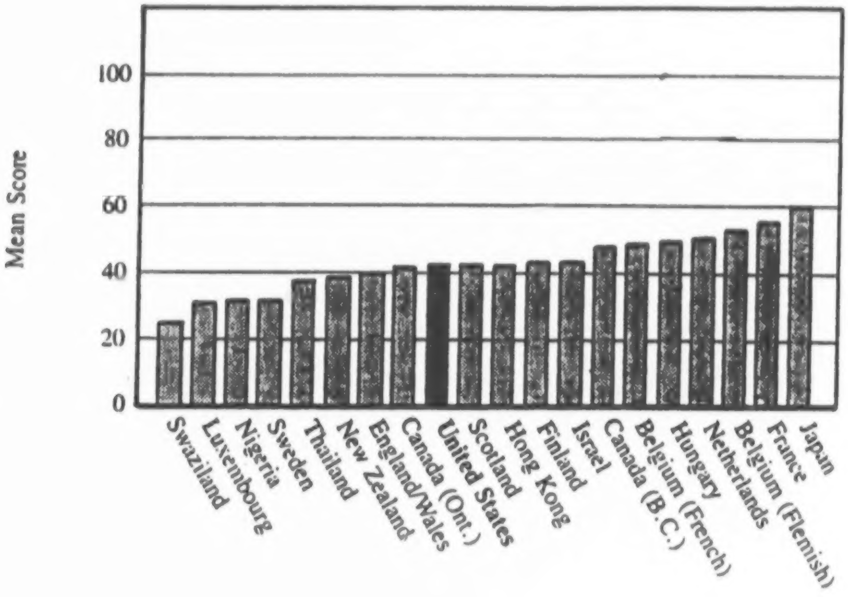


Figure 11 International Achievement in Algebra (Population A)

6. Yield

"Yield" may be defined in the context of this study as "what proportion of students has learned how much mathematics." From an international perspective, U.S. yield in mathematics is low. In Figure 12, data on the intended, implemented and attained curriculum are presented for five countries: Canada (British Columbia), Japan, England

and Wales, Sweden and the United States. The height of each bar graph is an index of the content of the curriculum as intended, implemented and attained. The width of the bar reflects the retentivity of the school system for the country. Therefore, the area of each bar may be thought of as a yield measure for the respective country.

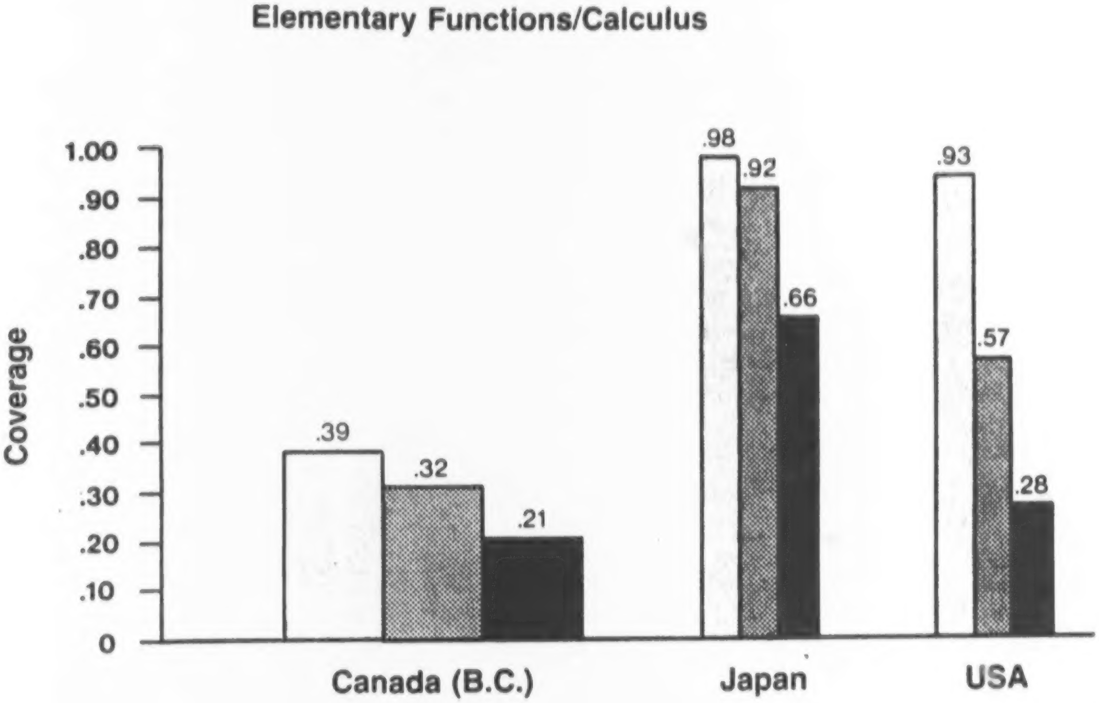


Figure 12 **Intended, Implemented and Attained Coverage in Population B Mathematics for Five Countries (Width of Bar Graph Reflects Population B Retentivity)**

(Figure 12)

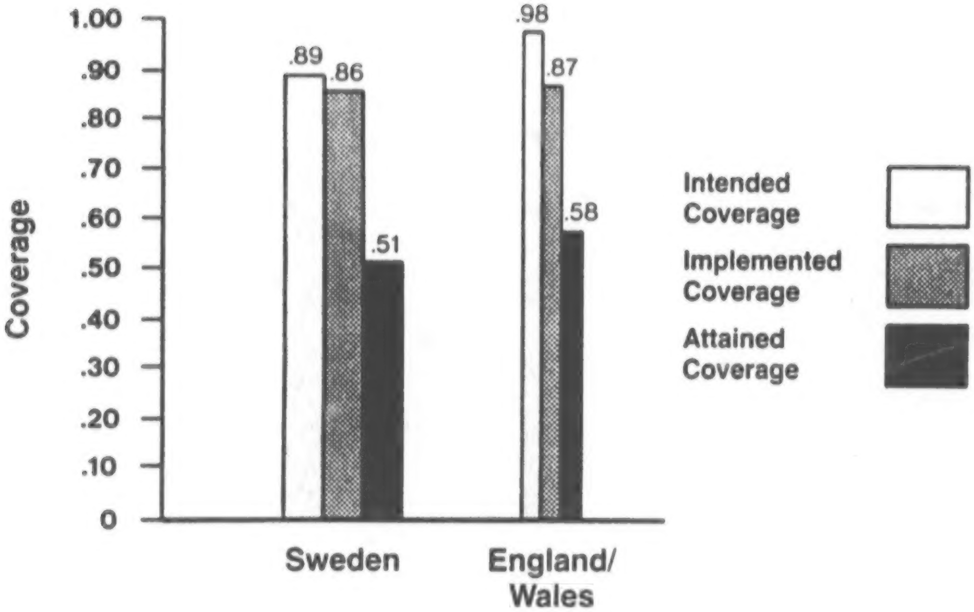

1.00
.90
.80
.70
.60
.50
.40
.30
.20
.10
0
Coverage
.89
.86
.51
.98
.87
.58
Sweden
England/
Wales
Intended Coverage
Implemented Coverage
Attained Coverage

Figure 12 indicates a generally negative association of retentivity with achievement. That is, less retentive (more selective) systems tend to have high achievement scores. The notable exception is Japan, with high achievement even though retentivity in Population B mathematics is comparable to that for Sweden and the United States.

7. Attitudes and opinions

7.1 Students

Overall, the attitudes of U.S. students toward mathematics tended to be positive. Students feel that the study of mathematics helps them to think logically and that the subject is a good one for creative people. They also seem to have positive self-concepts with respect to mathematics, and the majority report that they want to do well in the subject (as do their parents). These findings contrast favorably with those for Japanese students. Generally, in Japan, students have rather negative attitudes toward the subject (see Table 4).

As Kifer (1985) has commented, "Despite how well Japanese students did on the achievement test, compared with students in other systems they find this...mathematics content and activities more difficult and they like them less. They also ascribe only an average amount of importance to them" (page 11).

Table 4

Attitudes of Population A Students Toward Mathematics in School (from Kifer, 1985)

		Important	Easy	Like
		Nigeria		Nigeria
	20		Sweden	Swaziland
		Israel	Nigeria	
E		Thailand	Israel	Israel
			Swaziland	Thailand
F	10	9		
		16	6	
F		17, 18	3, 4, 9, 12	18
		6, 11	20	8, 9, 12, 16
E	0	8, 12, 15, 20	2	6, 7
		3, 4, 5, 7	7, 16	2, 3
C		13	8, 15	4, 5, 15
			13, 19	20
T	-10		5	13
		(French)	(Flemish)	(Flemish)
S		Belgium	Belgium	Belgium
		(Flemish)		
	-20	Belgium		
			Japan	Japan

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1 = (Flemish) Belgium
2 = (French) Belgium
3 = British Columbia
4 = Ontario
5 = England and Wales
6 = Finland
7 = France
8 = Hong Kong
9 = Hungary
10 = Israel
11 = Japan
12 = Luxembourg
13 = Netherlands
14 = Nigeria
15 = New Zealand
16 = Scotland
17 = Swaziland
18 = Sweden
19 = Thailand
20 = USA

7.2 Teachers

Teacher attitudes toward mathematics tended to be positive, with Population B teachers, as expected, exhibiting a more dynamic view of the topic than their Population A counterparts.

Interesting contrasts were found with teachers at corresponding levels in Japan. With respect to mathematics teaching, responses were obtained on the dimensions of importance of mathematics teaching, responses were obtained on the dimensions of importance of mathematics, ease in teaching the subject and how well teaching mathematics was liked. The Japanese teachers tended to regard mathematics teaching as somewhat more important than did the U.S. teachers. However, on the Easy and Like dimensions, the U.S. teachers were in the middle group while Japanese teachers were among the most negative.

Perception of Ease of Teaching

The vast majority of teachers from the United States reported that their mathematics classes were either fairly easy to teach or very easy to teach. In Japan on the other hand, between 30% and 40% found their classes hard to teach (see Table 5).

Table 5

Ease of Teaching Mathematics in General and to the Class Sampled as Rated by Eighth Grade Mathematics Teachers (Percent of Teachers: All Class Types Pooled)

Ease	In General		To the Sampled Class	
	US	J	US	J
Very Easy	46	1	25	0
Easy	43	8	43	10
Neutral	7	50	20	58
Hard	5	38	13	27
Very Hard	0	3	1	5

Reasons for Poor Achievement The Population A teachers were asked to select from a list those factors that they believe account for lack of satisfactory progress of their mathematics students. The teachers from the two countries were remarkably similar in their responses to the first three "student-oriented" reasons--lack of motivation, lack of ability and absenteeism. Japanese teachers contrast dramatically with U.S. teachers in that they more frequently attribute such factors as student misbehavior, student fears of mathematics, large classes and lack of time. The most dramatic contrast, perhaps, is that 1/3 of the Japanese teachers indicate their own lack of proficiency as an important reason for poor progress of their students. Only 3% of U.S. teachers cited this reason (see Table 6).

Table 6

Reasons for Lack of Satisfactory Progress by Students in the Sampled Classes as Rated by Eighth Grade Teachers (Percent of Teachers)

Reason	A Very Important Reason		Not An Important Reason	
	US	J	US	J
Student indifference or lack of motivation	51	51	11	8
Student lack of ability	45	38	14	12
Student absenteeism	39	35	35	30
Student misbehavior	12	22	55	30
Debilitating fear of mathematics	11	19	56	31
Too many students	10	30	62	20
Limited resources and materials	7	13	78	41
Insufficient school time allocated to mathematics	5	23	85	30
Insufficient proficiency on my part in dealing with students having the kinds of difficulties found in the target class	3	33	80	12

Goals for Mathematics Teaching Problem solving is, for U.S. teachers, the most highly ranked goal (from a stated list) for eighth grade mathematics, with developing an awareness of the importance of mathematics and establishing a basis of knowledge and skills as important goals, as well. For Japan, interest in mathematics is the most important goal. Problem solving ranks about in the middle range of importance ratings (see Table 7).

Table 7

Relative Importance of Goals in Teaching Mathematics as Rated by Eighth Grade Mathematics Teachers (Percent of Teachers)

Goal	Relatively More Important		Relatively Less Important	
	US	J	US	J
Develop a systematic approach to solving problems	63	34	5	11
Develop an awareness of the importance of mathematics in everyday life	61	18	7	20
Perform computations with speed and accuracy	57	48	8	12
Know mathematical facts, principles and algorithms	55	20	8	20
Become interested in mathematics	47	65	9	6
Develop an attitude of inquiry	38	32	5	9
Understand the logical structure of mathematics	30	13	17	27
Develop and awareness of the importance of mathematics in the basic and applied sciences	21	14	20	44
Understand the nature of proof	13	10	63	59

Importance of Resources According to U.S. teachers, the most important instructional resources are found in the textbook, in mathematical content recalled from course work and from tests. These factors are cited by Japanese teachers, too, but with less strength (or emphasis). It is interesting to note that relatively large proportions of teachers in both countries seem to discount the importance of external examinations, the syllabus and published visual aids as instructional resources for mathematics teaching (see Table 8).

Table 8

Importance of Resources Supporting Teaching as Rated by Eighth Grade Mathematics Teachers

Resource	Mean Importance Rating	Percent Rating as of Highest Importance		Percent Doing Without Resource Now	
		US	J	US	J
Published textbooks	3.14	49	21	2	22
"What you remember from mathematics courses you have taken"	3.09	45	7*	1	22*
"Tests you have written"	3.09	40	11	2	21
"Examples that you have made yourself"	3.09	38	5	4	30
"Problem sets you have written yourself"	2.71	24	3	5	28
Published workbooks and problem sets	2.07	13	3	13	21
"Advice you have received in the past year from other teachers"	2.00	8	2	7	32
"What you remember from education courses you have taken"	1.92	9	3	5	23
"Knowledge of what is on external exams taken by your students"	1.71	5	1	18	37
Official Syllabus	1.67	7	7	16	21
"Visuals (slides, transparencies, posters) you have made yourself"	1.69	7	2	25	31
"Advice received in the last year from administrators (e.g., department head, principal, curriculum supervisor)"	1.64	7	1	17	40
Published tests	1.53	8	1	23	20
Published visuals (slides, transparencies, posters)	1.20	3	0	35	51

* Note: Wording in Japanese questionnaire: "...remembering the way you were taught mathematics..."

8. Summary

Within the framework of SIMS, Japanese mathematics education looks very good. High proportions of students are learning a great deal of mathematics. It is noteworthy that this is taking place in the context of large classes, and only modest amounts of available instructional time. Furthermore, Japanese students (at Population A) were one year younger than those in most other countries.

Teacher coverage of the curriculum in Japan tends to be high, and with relatively low variation. Considerable variation in instructional strategies is in evidence. In short, the Japanese educational system appears to be very efficient.

In the U.S., by contrast, overall mathematics achievement is low. This is the case in spite of smaller classes and more allocated time for mathematics. Teachers exhibit a relatively limited repertoire of instructional approaches. However, the generally positive attitudes of both students and teachers toward mathematics in the U.S. is worthy of note.

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Japan

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Sawada, Toshio, Ed. Volume II: *Mathematics Achievement and Associated Factors of Secondary School Students. Tokyo: National Institute for Educational Research, March 1982.*

Sawada, Toshio, Ed. Volume III: Mathematics Achievement and Teaching Practice in Lower Secondary Schools (Grade 7) Tokyo: National Institute for Educational Research, March 1983.

United States

United States Summary Report: Second International Mathematics Study. Champaign, IL: Stipes Publishing Company, 1985.

Sweden

Murray, Asa and Robert Liljefors. Matematik i Svensk Skola. Skoloverstyrelsen, 1983.

Appendix I

United States Reports

United States Summary Report: Second International Mathematics Study. Champaign, IL: Stipes Publishing Company, 1985.

Technical Report I, Item Level Achievement and OTL Data, May 1985

Technical Report II, Questionnaire Data for Schools, Teachers and Students, (November 1985)

Technical Report III, Classroom Processes Data, (November 1985)

Technical Report IV, Instrument Book, Achievement Tests and Background Questionnaire, (December 1985)

Technical Report V, Instrument Book, Classroom Process Questionnaires (December 1985)

Detailed National Report, (August 1986)

The Under-Achieving Curriculum: Assessing U.S. School Mathematics from an International Point of View. (working title). (to appear, December 1986)

Classroom Processes in School Mathematics:
    Volume I : Eighth Grade
    Volume II: Advanced Mathematics

(Monographs of the Journal for Research in Mathematics Education, National Council of Teachers of Mathematics, to appear, Spring 1987)

(Note: The last two reports are funded by grants from the National Science Foundation.)

Discussion of Professor Travers' Paper

Hashimoto: On behalf of Japanese delegates, I request that American delegates speak English slowly like Professor Jerry Becker. Professor Travers has given us very interesting results through the IEA study. Now, are there questions or comments?

Travis: I was wondering how are these reports used in both Japan and the U.S.? Are they just informational or are they intended to effect curriculum change?

Travers: I'll let the Japanese go first.

Sawada: We have already published the national reports in three volumes based on the Japanese data. But the results of curriculum analysis of the Second IEA Mathematics Study has not yet been published.

Travers: I'll give an answer in a couple of ways. First of all, I cannot claim that this has happened by design, rather it happened more by happy coincidence that two of the major people in preparing the U.S. reports are, respectively, the past and the current President of the National Council of Teachers of Mathematics. I've asked all of the people to send in to me reports of what they have been doing about the findings; i.e., how they have been using it. From those two gentlemen, Professors Joe Crosswhite and John Dossey, I have an impressive list of the numbers of talks they've given around the country and it runs into almost 100 talks and many thousands of teachers they have reached. I am not sure how one assesses what use is being made in terms of implications, but that's at least getting the word out so that people are aware of some of these findings. Now, one of the things about the United States that many people around the world find curious is the lack of a kind of a centralized thrust in terms of education. By tradition, education is very much a local matter. So the Department of Education in Washington, D.C. is, at best, regarded with a certain amount of suspicion in terms of making pronouncements. So one does not look to a central group like the U.S. Department of Education for policy actions. But, in the case of mathematics education, a very satisfactory alternative has been the appointment of a group called the Mathematical Sciences Education Board. It is a board that has been established to, among other things, look at issues of policy in mathematics education. As it turns out, the board is very interested in the findings of the study and is at the moment putting together a symposium that will be held in late fall or early winter in Washington to examine the findings and to make policy recommendations based on the findings. That's another kind of thing that is being done.

Shimada: The project was a theoretical one and focused on fact-finding. To interpret the implications of the results for educational decision making is left to the outsiders who have an interest in the study. Its implication may be interpreted in many ways according to different viewpoints. The IEA study gathered many data but not all will be used in the final report. One important feature of the IEA is that after all processes are completed, all data will be internationally opened to those who want to analyse the data from different viewpoints.

Speaking from the Japanese experience in the IEA First Mathematics Study, the reports were published as books by the National Institute for Educational Research, and became gradually known to all teachers and scholars in the field. For a variety of reasons it induced many different opinions regarding the state of mathematics education in those days. Those criticisms or considerations reflected on deciding some action in mathematics education. So its effect is indirect. What I referred to in the discussion of my paper is one example.

Rachlin: You had mentioned earlier, Mr. Hashimoto, that Dr. Travers has just finished a new edition of a methods text, the secondary methods text and, as an author for a methods text, he originally wrote his first version before the international study began and a new version is coming out in the midst of the international study. Professor Travers, are there changes that you have made and suggestions to teachers for the way that they teach based on what you have learned from the international study?

Travers: Unfortunately, part of the information you have is not correct - we have not finished the second edition yet. I say that also in self-defense because I haven't really thought through all the implications, but I am sure there will be some. Let me give you one. I won't take the time to go to the overhead, but let me remind you of some of those displays that show incredible relationship from a statistical point of view between what teachers teach and what kids learn. I think through the research on teaching some of us have grown rather defensive about the fact that we don't have a lot of evidence out there, that is, hard data that there is much of a relationship between what teachers teach and what kids learn. We have in the IEA data indisputable evidence that this is the case. To those of us who make our living by teaching this should not be a surprise, but reassuring. I think one of the points I would like to see made much more strongly is that you have to take the teaching act very seriously. If you've got an objective make sure you cover it. A sort of necessary but not sufficient condition for kids learning the stuff is that, by golly, you make sure you get to it in your program. There is another one that is terribly profound and I don't know how to deal with it yet. I think we are making a terrible mistake in the United States by an early differentiation of kids into various groups. Someone has said we have them not only in different cafeteria lines but they are in different buildings, essentially. I think I will let that second one just hang, because the implications are profound and I think it is something about which our Japanese friends are probably puzzled because in Japan you just don't understand the American situation: that is, we are convinced in the United States that the way in which to teach effectively and to help kids develop their highest potential is to put them into different kinds of classses early in the school. I'd be happy if the Japanese would like to comment on that.

Sugiyama: We want to do it, but because of the Native Japanese people they cannot do it, nor are allowed to do it. Japanese society is a democratic one, but it is the so-called "democratic on the surface." It is not truly democratic. I think that Americans seem to give equal opportunity for growth. The Americans may make differentiations, but that's because they want to give everyone equality in growth.

Travers: Well, my comment is that many of our kids are not getting an opportunity to grow in algebra, for example - they are not being taught algebra.

Whitman: Professor Travers, of all the countries in the study, what countries besides Japan were heterogeneously grouped?

Travers: That is a little hard to determine factually. We can ask countries what they are doing, but we wanted to do this by looking at the data. The way we decided to handle it was by sampling two classrooms per school so that we could get a measure of how much within-school variation there was in achievement. And on the basis of that it looks like, for example, in the European tradition, there is a lot of differentiation because it is part of the tradition that more able kids go to the gymnasium, so I'm not holding that up as a model. I think that would probably be even a worse situation than some. From a democratic pont of view I think that's repugnant. In terms of what you say, for example, as official policy, British Columbia and Ontario say they do not group by ability. But it was very interesting when we looked at the opportunity to learn data where one of the questions is "Have you taught this? If not, why not?" A lot of the British

Columbia teachers were checking the response "We don't teach it for other reasons" and the people out there said that other reason is because they are doing de facto grouping. However, it is against ministry policy, so they don't want to state it.

Silver: Ken, you mentioned in discussing one of the items that the pattern of gender differences was similar across all countries. In general, comparing the Japanese and U.S. performance on items for which there are gender differences, do you find the same pattern cross-culturally or is that something that bears looking at in a study?

Travers: The matter of gender differences is just not receiving attention. There is a group in Ontario that is working on this and I'm trying to recall what they found. I mentioned this one because it is one of the classic kinds of items on spatial visualization but my recollection is that, by and large, we do not find striking patterns of gender differences across the countries and one would have hoped that, because of cultural kinds of differences, one would find gender differences to correspond with that and we haven't found any - that's my recollection. It certainly bears further study.

Sawada: I don't think teachers have responded that there are differences in gender. I think two things so it is a very large difference gender between two countries. Japanese gender differences are large.

Travers: Achievement, yes, I was referring to achievement.

Sugiyama: In Japan, students want to be equal or to be not different from each other, so they are much concerned with low achievers. Much time is used to teach lower level students, and it might happen that students of high achievement do not have opportunites to grow enough.

Travers: I would point out that one of the most important analyses that has been done is to look at the achievement of the top one percent of the students across all of the countries. That helps to deal with differences in retentivity, for example, and we found that on that one Japan wins nicely and the U.S. loses.

I was reading some of the papers that came prior to this conference. There was something in one of the papers by Professor Azuma, formerly of Tokyo University, on views that parents in the U.S. and Japan have about their children's achievement in school. I found a striking correspondence between that and my table on page 175. Here we ask the teachers how easy they found mathematics teaching to be. By and large, American teachers found it really very easy and Japanese teachers find it very hard. I think clearly one of the issues has got to be this business of ability grouping. If you have all of these kids of various abilities in one class, that's certainly one factor that makes the teaching of mathematics very hard in Japan. But overall it is kind of an interesting cultural comparison.

Hashimoto: I'm sorry, but time is nearly up, but we'll take one more question.

Silver: Just as a follow-up to that, it may be that Japanese teachers find it hard because they are really teaching mathematics and U.S. teachers aren't.

Travers: Because they have had courses in the pedagogy of mathematics.

Hashimoto: Now time is up. We have had useful discussion in this session. Thank you very much.

Professor Clarkson's Paper

Nohda: It is my pleasure to introduce Dr. Sandra Clarkson. She is Associate Professor of Mathematics at Hunter College, City University of New York. She has worked as a junior high school teacher, a teacher of teachers and currently is Director of the Mathematics Learning Center at Hunter College. She is an author of several books and served last year as an advisor and consultant to a company developing an elementary school mathematics textbook series.

Dr. Clarkson arrived here on Sunday night, when we were enjoying the reception at the Hotel. After a long trip from London via New York, she came here to speak in this session.

PROBLEM SOLVING INSTRUCTION AND THE CHARACTERISTICS OF WORD PROBLEMS IN ELEMENTARY SCHOOL TEXTBOOK SERIES IN THE UNITED STATES

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Note: The purpose of this paper is to provide some descriptive information of elementary mathematics textbooks in the United States. The emphasis is on word problems - the operations or strategies required to solve them, their context, length, size of numbers, and density. The "counts" of problems are not intended to be absolute numbers but to give a "ball park figure" to use in comparison from one text to another. Every attempt was made by this author to be consistent in the analysis from one text to another and in the identification and classification of word problems.

One of the most reliable predictors of the content taught in the elementary school classroom is the content found in the elementary school textbook. Lacking a clear curriculum guide or a strong mathematics background and lacking ancillary and manipulative materials, most elementary school teachers in the United States rely on the content and approaches found in their grade level mathematics textbook. What, then, can we conclude about instruction in problem solving?

In 1981, it was reported that curricula for the primary level contained "practically no mathematical problem-solving experiences " (Greenes, 1981). Most of the major textbook publishers in the United States now claim to contain much problem solving, even beginning in Kindergarten. To determine what extent problem solving is included, the existing scope and sequence charts of 10 major publishers (Appendix I) were surveyed to determine the strategies taught in each program for grades three, five, and seven (see Tables 1 - 3). A lack of consensus was found as to what constitutes problem solving.

The number of different strategies taught within a grade level differ from series to series and grade to grade. In the third grade (Table 1), from one to twelve different strategies are taught, depending on the text. Text series I, C, and J teach 12, 11, and 10 strategies, respectively. The most commonly taught strategies for this grade level are "choosing correct operations," "solving problems with too much information," and "drawing/using diagrams." For grade 5 (Table 2), the number of different strategies taught ranges from 13 to 4, with series A, C, and E teaching the largest number (13, 12, 11). The most commonly taught strategies were "solving problems with too much information" and "solving multi-step problems." In grade seven (Table 3), text series C, D, and F taught the most different strategies (13, 12, 10). The least number taught was 5. The most commonly taught strategies were "solving problems with too little information" and "using/drawing diagrams."

For the most part, the text series showed a lack of consistency in the strategies taught from year to year (Table 4). There were only two strategies that were taught in more than half the text series for all three of the selected grades (3, 5, & 7). Those two strategies were "solving problems with too much information" and "using/drawing a diagram." Looking at the individual text series, we find that from 1 to 11 strategies were taught in all three years. Text series C taught 11 strategies. The next closest were series G and J with 6 strategies.

With the teaching of all these strategies, what kinds of word problems are being included in the textbooks? Five major textbook series (1985-87 copyright) were chosen and the word problems in the third and fifth grades were analyzed in the following categories: problem characteristics, including the use of key words, the use of strategies necessary for solving a problem, problem density, context (fantasy, realistic, factual), and difficulty.

A major factor in the ability to solve story problems in Arithmetic is a student's reading ability (Ballew & Cunningham, 1982). Textbook publishers have tried presenting problems with fewer words (Eicholz, O'Daffer & Fleenor, 1978); however, this format has not been shown to improve the problem solving ability of students in the third through seventh grades. In fact, the traditional format appears to be easier to interpret, especially for students of high reading ability (Moyer, Moyer, Sowder & Threadgill-Sowder, 1984). One aspect of reading ability that may aid in problem solving performance is the recognition of certain "key words" that may alert students to the operation needed in a problem.

There is some disagreement about the usefulness of "key words" in teaching students to solve word problems (Nesher & Teubal, 1975). Key words are words that suggest the operation needed to solve a word problem, words like "in all," "sum," "difference," "have left" (see Table 5) (Caldwell, 1984). To determine the use of key words in the textbooks, problems were classified according to the operations needed to solve them and whether or not there were key words in the text of the problems. These results are reported in Tables 6 - 9.

Word problems that use the operation of addition in Grade 3 (Table 6a) and Grade 5 (Table 6b) textbooks rely on key words only 40% of the time. The most commonly used key words are "in all" and "total" for grade 5, and "in all," "together" and "more" for grade 3. For subtraction (Tables 7a and 7b), the key words "change" and the two phrases "have left" and "how many more" were clearly most frequently used. Key words appeared in 54% of the subtraction problems for the third grade and in 65% for the fifth grade. For multiplication (Tables 8a and b) key words were used 63% of the time for grade 3 and 55% for grade 5. The most common key words were "times" and "each." Division (Tables 9a and b) also relied heavily on the use of the key word "each." Clearly, this word loses its effect as a hint when it is used equally as often for both multiplication and division. Key words were used in 73% of the problems in grade 3 and 55% in grade 5. Many of the problems were multi-step problems (Tables 10a and 10b), using any combination of operations for solution. As a matter of observation, most of these contained key words that would aid in their solution. The number of multi-step problems in a text ranged from 18 to 39 for grade 3 and from 33 to 175 for grade 5.

Use of Problem Solving Strategies Problems require a problem solving strategy, in place of, or together with, one or more operations. The emphasis on problem solving strategies in the textbooks followed the release of a set of recommendations for teaching mathematics issued by the National Council of Teachers of Mathematics (NCTM) and the National

Council of Supervisors of Mathematics (NCSM). The Position Paper on Basic Skills released in 1977 included strong recommendations to include the teaching of general heuristics like the following: decide what information is given, or needed; make a diagram; write a mathematics sentence using symbols to show operations; make a table, graph, and/or chart; look for a pattern; try a formula; work backwards; guess and check--estimate; think of a simpler problem; think of a similar problem; try to solve the problem several ways. This suggestion, together with the statement that "mathematics education must not emphasize computational skills to the neglect of other critical areas of mathematics" came as a response to the decreasing test scores, especially in applications and problem solving, which accompanied a strong "Back to Basics" movement in the United States in the seventies.

Texts for grades 3 and 5 were analyzed to find the most often needed problem solving strategies. For this analysis, the problem solving strategies taught in any of the texts for grades 3 and 5 were listed and then a count was made of the problems that needed such strategies for solution. The most commonly needed strategies in the third grade were, in order, "Guess and Check," "Make a Table," "Missing Information," and "Draw a Picture." The total number of problems needing any problem solving strategies in each of the three text series A, B, and D were 33, 30, and 8. For grade 5, the use of strategies was more prevalent. The most commonly used strategies were, in order, "Guess and Check," "Draw a Picture," "Make an Organized List," and "Missing Information." The total number of problems needing problem solving strategies in a series ranged from 0 - 85. By the seventh grade, students have been exposed to many different strategies. To find the number of available problems that could use a given strategy (Table 15), all strategies labeled "Problem Solving" were listed with the total number of problems included for each one. No attempt was made here to determine the necessity for a given strategy, this is merely a count of the number of problems available for practice with the strategy.

Although most text series now claim to teach problem solving strategies, the actual problem solving required is often minimal. Despite the instruction in problem solving strategies, the majority of problems given in the selected five textbook series require no such strategies to solve them. And, in the comparison of any two instructional sets, great variation will be seen in the types of problems given and the approaches needed to solve them.

The newest texts, copyrighted in 1987, show much more emphasis on problem solving strategies than do their 1985 and earlier counterparts. They do include problems that can be solved only with the use of such strategies. This emphasis has resulted from the issuance of the California State Department of Education's Mathematics Framework for California Public Schools. This framework is California's statement of what it expects from all the mathematics textbooks adopted in California. The influence of California will be seen in all mathematics textbooks copyrighted in 1987 and, perhaps, beyond that.

There are several additional factors that may account for a student's ability or inability to solve verbal problems. These factors are problem density (how many problems are there to spread over the school year); problem context (whether the problems are factual (with real data), realistic, but not factual; or fantasy-like); and problem difficulty. Analyses for all these characteristics were done of 3rd and 5th grade texts.

Problem Density was determined by dividing the total number of "word problems" by the approximate number of school days in the United States to find the average number of problems per day available to the teachers of each of the text series. Table 12a shows the problem density is about the same for each of the third grade texts examined. Table 12b shows that the density for the fifth grade texts ranges from 1.1 to 3.0. Most of these problems appear in groups throughout certain, but not all, chapters.

Problem Context was determined by analyzing the first three problems appearing after pages 25, 50, 75, 100, etc., to determine whether they were factual (contained verifiable information), realistic but not factual (possible situations and numbers that are also possible, but not true situations), or fantasy (made up situations and data; not realistic). For the third and the fifth grade textbooks (Table 13), the vast majority of the problems are realistic, but not factual.

Problem Difficulty was determined by analyzing the fifth word problem that appeared after pages 25, 50, 75, 100, etc., to find the problem length, size of the numbers in the problem, the operation(s) needed to solve the problem, and the order of the numbers needed to solve the problem. This information is included in Tables 14a-14h. The average number of words in each problem ranges from 19.1 words to 25.8.

My intention in this paper is to describe the word problems available in the mathematics textbooks for elementary grades three, five, and seven. In this way, I hope to give what I consider a relatively true picture of the classroom use of word problems in the United States. However, whether or not the teachers are capable of presenting problem solving to their students using the materials available in the textbooks cannot be answered by simply analyzing these texts.

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TABLE 1 Scope and Sequence Information: Heuristics Used in Grade 3 Textbooks

TEXT SERIES	A	B	C	D	E	F	G	H	I	J
PROBLEM SOLVING STEPS	3	3	3	-	-	-	-	-	3	3
CHOOSING CORRECT OPERATIONS	3	3	3	3	3	3	3	-	3	3
ASKING QUESTIONS/FORMULATING PROBLEMS	3	-	3	3	3	-	-	-	3	-
PROBLEMS WITH TOO LITTLE INFORMATION	3	3	3	3	-	-	3	-	3	3
PROBLEMS WITH TOO MUCH INFORMATION	3	3	3	3	3	3	3	-	3	-
MULTI-STEP PROBLEMS	3	3	3	-	-	-	-	-	3	3
ORGANIZING INFORMATION	-	-	3	3	-	3	3	-	3	3
IDENTIFYING REASONABLE ANSWERS	-	3	-	-	3	3	-	-	-	3
USING/DRAWING DIAGRAMS	3	3	3	3	-	3	3	-	3	3
SOLVING A SIMPLER PROBLEM	-	-	-	3	-	-	-	-	3	-
USING/FINDING PATTERNS	-	-	3	-	-	3	-	-	3	3
USING FORMULAS AND EQUATIONS	-	-	-	-	3	3	-	-	-	-
GUESS AND CHECK	3	3	-	-	-	-	-	-	3	-
WORKING BACKWARDS	-	-	-	-	-	-	-	-	-	-
MAKING AN ORGANIZED LIST	-	-	3	-	-	-	-	-	-	3
USING LOGICAL REASONING	-	-	3	3	-	-	-	3	3	3

TABLE 2 Scope and Sequence Information: Heuristics Used in Grade 5 Textbooks

TEXT SERIES	A	B	C	D	E	F	G	H	I	J
PROBLEM SOLVING STEPS	5	5	5	-	5	-	-	-	-	5
CHOOSING CORRECT OPERATIONS	5	5	5	-	5	5	5	-	-	5
ASKING QUESTIONS/FORMULATING PROBLEMS	5	-	5	-	5	5	-	5	5	-
PROBLEMS WITH TOO LITTLE INFORMATION	5	5	5	5	5	5	5	5	5	-
PROBLEMS WITH TOO MUCH INFORMATION	5	5	5	5	5	5	5	-	5	-
MULTI-STEP PROBLEMS	5	-	5	5	5	5	5	5	5	5
ORGANIZING INFORMATION	5	-	5	5	-	5	5	-	5	5
IDENTIFYING REASONABLE ANSWERS	5	-	-	5	5	5	-	5	-	5
USING/DRAWING DIAGRAMS	5	-	5	5	5	5	5	-	5	-
SOLVING A SIMPLER PROBLEM	5	5	5	-	5	-	-	-	5	-
USING/FINDING PATTERNS	5	-	5	-	-	5	-	-	-	5
USING FORMULAS AND EQUATIONS	-	-	-	5	5	5	-	-	-	-
GUESS AND CHECK	5	5	-	-	5	-	-	-	-	-
WORKING BACKWARDS	5	5	-	-	-	-	-	-	-	5
MAKING AN ORGANIZED LIST	-	-	5	-	-	-	-	-	-	-
USING LOGICAL REASONING	-	-	5	5	-	-	-	-	5	5

TABLE 3 Scope and Sequence Information: Heuristics Used in Grade 7 Textbooks

TEXT SERIES	A	B	C	D	E	F	G	H	I	J
PROBLEM SOLVING STEPS	7	7	7	-	7	-	-	-	-	7
CHOOSING CORRECT OPERATIONS	-	7	7	7	-	7	7	7	-	-
ASKING QUESTIONS/FORMULATING PROBLEMS	7	-	7	7	7	7	-	-	-	-
PROBLEMS WITH TOO LITTLE INFORMATION	7	7	7	7	7	7	7	7	-	7
PROBLEMS WITH TOO MUCH INFORMATION	-	7	7	7	7	7	7	7	-	7
MULTI-STEP PROBLEMS	7	-	7	7	7	7	7	7	-	7
ORGANIZING INFORMATION	-	-	7	7	-	-	7	-	7	7
IDENTIFYING REASONABLE ANSWERS	7	-	-	7	7	7	-	7	-	7
USING/DRAWING DIAGRAMS	7	7	7	7	7	7	7	7	7	-
SOLVING A SIMPLER PROBLEM	7	7	7	-	-	7	-	-	-	-
USING/FINDING PATTERNS	7	7	7	7	-	7	-	-	7	7
USING FORMULAS AND EQUATIONS	7	7	7	7	7	7	7	7	-	-
GUESS AND CHECK	7	7	-	-	7	7	-	-	7	-
WORKING BACKWARDS	-	7	-	-	-	-	-	-	-	-
MAKING AN ORGANIZED LIST	-	-	7	7	-	-	-	-	-	-
USING LOGICAL REASONING	-	-	7	7	-	-	-	-	7	7

TABLE 4 Scope and Sequence Information: Heuristics Used in Grades 3, 5, and 7

TEXT SERIES	A	B	C	D	E	F	G	H	I	J
PROBLEM SOLVING STEPS	X	X	X							X
CHOOSING CORRECT OPERATIONS		X	X			X	X			
ASKING QUESTIONS/FORMULATING PROBLEMS	X		X		X					
PROBLEMS WITH TOO LITTLE INFORMATION	X	X	X	X			X			
PROBLEMS WITH TOO MUCH INFORMATION		X	X	X	X	X	X			
MULTI-STEP PROBLEMS			X		X		X	X		
ORGANIZING INFORMATION			X	X			X		X	X
IDENTIFYING REASONABLE ANSWERS					X	X				X
USING/DRAWING DIAGRAMS	X		X	X		X	X		X	
SOLVING A SIMPLER PROBLEM										
USING/FINDING PATTERNS			X			X				X
USING FORMULAS AND EQUATIONS					X					X
GUESS AND CHECK	X	X								
WORKING BACKWARDS										
MAKING AN ORGANIZED LIST			X							
USING LOGICAL REASONING			X	X					X	X

TABLE 5

KEY WORDS

ADDITION

sum
in all
together
total
plus
more (than)
greater (than)
increased
add
rise
gain(ed)
earn
save

SUBTRACTION

difference
reduced (by)
less (than)
decreased
minus
fewer (than)
remain
subtract
fall
lost
take away
spend
change

MULTIPLICATION

product
times
multiply
each

DIVISION

quotient
divide
per
each

TABLE 6A - ADDITION USE OF KEY WORDS IN VERBAL PROBLEMS 3RD

	A	B	D
SUM			
IN ALL	10	50	11
TOGETHER	13	8	4
TOTAL	6		1
PLUS			
MORE (THAN)	4	3	19
GREATER (THAN)			
INCREASED			
ADD			4
RISE			
GAIN(ED)			
EARN	1		
SAVE			
NO KEY WORDS 3 ADDENDS	17	23	10
NO KEY WORDS	70	42	36
TOTAL	121	126	85

TABLE 6B - ADDITION USE OF KEY WORDS IN VERBAL PROBLEMS

	A	B	C	D	E
SUM				1	
IN ALL	5	25	4	2	4
TOGETHER	7		3		2
TOTAL	18	4	3	2	1
PLUS					
MORE (THAN)	1	1		12	
GREATER (THAN)					
INCREASED					
ADD	1	1			1
RISE					
GAIN(ED)					
EARN					
SAVE					
NO KEY WORDS 3 ADDENDS	6	3	6	6	7
NO KEY WORDS	19	13	9	49	29
TOTAL	56	47	25	72	44

TABLE 7A - SUBTRACTION USE OF KEY WORDS IN VERBAL PROBLEMS

	A	B	D
DIFFERENCE	1		
REDUCED (BY)			
SPEND			
DECREASED			
MINUS			
FEWER (THAN)	6		4
REMAIN			
SUBTRACT	1		7
FALL			
LOST	1		
TAKE AWAY			1
CHANGE	2	11	3
HAVE LEFT	13	28	12
HOW MANY MORE	28	27	33
NO KEY WORDS	61	26	64
TOTAL	113	92	124

TABLE 7B - SUBTRACTION USE OF KEY WORDS IN VERBAL PROBLEMS GRADE 5

	A	B	C	D	E
DIFFERENCE	1	1		5	3
REDUCED (BY)	6				
SPEND					
DECREASED					
MINUS					
FEWER (THAN)				6	
REMAIN					
SUBTRACT					
FALL					
LOST					
TAKE AWAY					
CHANGE	10	2	11	3	1
HAVE LEFT	6	10	6	1	2
HOW MANY MORE	16	32	9	18	23
NO KEY WORDS	18	16	10		47
TOTAL	57	61	36	33	76

TABLE 8A - MULTIPLICATION USE OF KEY WORDS IN VERBAL PROBLEMS

	A	B	D	
PRODUCT				
TIMES	2	3	13	
MULTIPLY	2			
EACH	35	59	39	
AREA				
NO KEY WORD	31	23	76	
TOTAL	70	85	88	
				63%

TABLE 8B - MULTIPLICATION USE OF KEY WORDS IN VERBAL PROBLEMS GRADE 5

	A	B	C	D	E
PRODUCT	1				
TIMES	4	15	6	7	6
MULTIPLY					
EACH	27	48	7	32	15
AREA	1	4	1		6
NO KEY WORD	30	33	28	65	51
TOTAL	61	100	42	104	78

TABLE 9A - DIVISION USE OF KEY WORDS IN VERBAL PROBLEMS

	A	B	D
QUOTIENT			
DIVIDE			
PER			
EACH	37	45	27
NO KEY WORD	11	14	15
TOTAL	48	59	42

TABLE 9B - DIVISION USE OF KEY WORDS IN VERBAL PROBLEMS GRADE 5

	A	B	C	D	E
QUOTIENT	1				
DIVIDE	4				
PER	5	1		2	1
EACH	25	76	23	18	9
NO KEY WORD	30	23	33	28	20
TOTAL	65	100	56	48	30

TABLE 10A GRADE 3

	A	B	D
MULTISTEP PROBLEMS	39	36	18

TABLE 10B

	A	B	C	D	E
MULTISTEP PROBLEMS	157	175	33	110	75

TABLE 11A PROBLEMS NEEDING IDENTIFIED HEURISTICS GRADE 3

	A	B	D	
GUESS AND CHECK	10	1	6	17
DRAW A PICTURE	6	2	2	10
MAKE A TABLE	2	12		14
MAKE AN ORGANIZED LIST	7			7
USE LOGICAL REASONING	3			3
WORK BACKWARDS				
SOLVE A SIMPLER PROBLEM				
FIND A PATTERN	5			
USE A FORMULA				
MISSING INFORMATION		12		12
TOTAL	33	30	8	

TABLE 11B PROBLEMS NEEDING THE IDENTIFIED HEURISTICS

	A	B	C	D	E	
GUESS AND CHECK	7	2			25	34
DRAW A PICTURE	12				16	28
MAKE A TABLE	2				12	14
MAKE AN ORGANIZED LIST	9				16	25
USE LOGICAL REASONING	5					5
WORK BACKWARDS	5				10	15
SOLVE A SIMPLER PROBLEM	2					2
FIND A PATTERN	10					10
USE A FORMULA	7				6	13
MISSING INFORMATION		12		12		24
REDUCED WORD COUNT - MAKING CHANGE						
REDUCED WORD COUNT - EST, MONEY MULT						
OTHER HEURISTIC						
TOTAL	59	14	0	12	85	

TABLE 12A PROBLEM DENSITY GRADE 3

	A	B	D
TOTAL PROBLEMS	424	428	373
PROBLEM DENSITY	2.4	2.4	2.1

TABLE 12B PROBLEM DENSITY

	A	B	C	D	E
TOTAL PROBLEMS	463	536	192	284	392
PROBLEM DENSITY	2.6	3.0	1.1	1.6	2.2

TABLE 13 - PROBLEM CONTEXT

		T	R	F	TOTAL PAGES
GRADE 3					
	A	3	35	0	358
	B	0	39	0	340
	C	0	29	0	295
GRADE 5					
	A	4	43	0	409
	B	3	42	0	382
	C	0	39	0	344
	D	0	27	3	313
	E	5	36	2	400

T-TRUE/FACTUAL
R-REALISTIC
F-FANTASY

TABLE 14A - PROBLEM DIFFICULTY TEXT A

	L	S	O	E
25	12	S	O	X
50	31	L	O	C
75	21	D	O	C
100	20	S	O	C
125	33	S	M	C
150	26	L	M	X
175	9	D	O	X
200	49	S	M	X
225	28	D	M	C
250	23	D	O	C
275	22	L	M	X
300	41	D	O	C
325	14	D	H	X
350	21	D	M	X
375	29	D	M	X
400	34	S	M	C
TOTAL PAGES: 409	25.8			
RANGE: 9 - 49				

L - WORD ORDER
S - NUMBER SIZE
    S - 0 - 99
    L - OVER 99
    D - FRAC/DEC/PCT

O - O - ONE OPERATION
    M - MORE THAN ONE
    H - HEURISTIC NEEDED

E - C - NUMBERS AND OPERATIONS FLOW SMOOTHLY IN ORDER
    X - MIXED ORDER OR NO CLEAR ORDER

TABLE 14B - PROBLEM DIFFICULTY   TEXT B

	L	S	O	E
25	31	L	O	C
50	26	S	O	C
75	27	D	O	C
100	20	D	O	X
125	23	L	O	C
150	53	S	H	X
175	14	D	O	C
200	26	D	M	C
225	20	D	O	C
250	20	S	M	X
275	23	D	O	X
300	27	S	O	X
325	12	D	O	C
350	21	S	M	C
375	26	D	M	C
TOTAL PAGES: 382	25.2			
RANGE: 14 - 53				

L - WORD LENGTH

S - NUMBER SIZE
S - 0-99
L - OVER 99
D - FRAC/DEC/PCT

O - O - ONE OPERATION
M - MORE THAN ONE
H - HEURISTIC NEEDED

E - C - NUMBERS AND OPERATIONS FLOW SMOOTHLY IN ORDER
X - MIXED ORDER OR NO CLEAR ORDER

TABLE 14C - PROBLEM DIFFICULTY TEXT C

	L	S	O	E
25	20	S	O	C
50	28	S	M	C
75	22	S	O	C
100	24	L	O	C
125	20	S	O	C
150	28	D	O	C
175	30	D	M	C
200	21	D	M	C
225	10	S	M	C
250	27	D	O	C
275	22	D	O	C
300	20	S	O	C
325	18	D	M	C
TOTAL PAGES: 344	22.3			
RANGE: 10 - 30				

L - WORD LENGTH

S - NUMBER SIZE
S - 0-99
L - OVER 99
D - FRAC/DEC/PCT

O - O - ONE OPERATION
M - MORE THAN ONE
H - HEURISTIC NEEDED

E - C - NUMBERS AND OPERATIONS FLOW SMOOTHLY IN ORDER
X - MIXED ORDER OR NO CLEAR ORDER

TABLE 14D - PROBLEM DIFFICULTY TEXT D

	L	S	O	E
25	10	L	M	X
50	24	S	M	C
75	15	L	O	C
100	17	S	O	C
125	-	-	-	-
150	15	D	O	C
175	22	D	O	C
200	33	D	O	C
225	20	D	O	C
250	20	D	O	C
275	15	S	O	C
TOTAL PAGES: 313	19.1			
RANGE: 10 - 33				

L - WORD LENGTH

S - NUMBER SIZE
  S - 0-99
  L - OVER 99
  D - FRAC/DEC/PCT

O - O - ONE OPERATION
  M - MORE THAN ONE
  H - HEURISTIC NEEDED

E - C - NUMBERS AND OPERATIONS FLOW SMOOTHLY IN ORDER
  X - MIXED ORDER OR NO CLEAR ORDER

TABLE 14E - PROBLEM DIFFICULTY TEXT E

	L	S	O	E
25	29	L	H	X
50	21	D	M	C
75	19	S	H	X
100	10	S	H	C
125	31	S	O	C
150	41	D	O	C
175	36	S	H	X
200	17	D	O	X
225	27	S	M	C
250	17	D	M	C
275	22	D	O	C
300	11	D	O	C
325	12	D	M	C
350	32	D	M	C
TOTAL PAGES: 400	23.2			
RANGE: 11 - 41				

L - WORD LENGTH

S - NUMBER SIZE
  S - 0-99
  L - OVER 99
  D - FRAC/DEC/PCT

O - O - ONE OPERATION
  M - MORE THAN ONE
  H - HEURISTIC NEEDED

E - C - NUMBERS AND OPERATIONS FLOW SMOOTHLY IN ORDER
  X - MIXED ORDER OR NO CLEAR ORDER

TABLE 14F - PROBLEM DIFFICULTY TEXT A, 3RD GRADE

	L	S	O	E
25	-	-	-	-
50	32	S	O	C
75	9	L	O	C
100	12	L	O	C
125	19	D	O	X
150	17	S	O	C
175	14	S	O	C
200	-	-	-	-
225	16	S	O	C
250	13	S	O	C
275	28	S	H	X
300	26	S	O	C
325	8	D	O	X
TOTAL PAGES: 358	17.6			
RANGE: 9 - 32				

L - WORD LENGTH

S - NUMBER SIZE
  S - 0-99
  L - OVER 99
  D - FRAC/DEC/PCT

O - O - ONE OPERATION
  M - MORE THAN ONE
  H - HEURISTIC NEEDED

E - C - NUMBERS AND OPERATIONS FLOW SMOOTHLY IN ORDER
  X - MIXED ORDER OR NO CLEAR ORDER

TABLE 14G - PROBLEM DIFFICULTY TEXT B, 3RD GRADE

	L	S	O	E
25	14	S	O	C
50	20	S	O	C
75	25	S	O	C
100	28	L	O	C
125	16	D	O	X
150	21	S	O	C
175	19	S	O	C
200	22	S	O	C
225	30	S	O	C
250	21	S	O	C
275	30	S	O	C
300	21	D	O	C
325	26	S	O	C
TOTAL PAGES: 340				

L - WORD LENGTH

S - NUMBER SIZE
S - 0-99
L - OVER 99
D - FRAC/DEC/PCT

O - O - ONE OPERATION
M - MORE THAN ONE
H - HEURISTIC NEEDED

E - C - NUMBERS AND OPERATIONS FLOW SMOOTHLY IN ORDER
X - MIXED ORDER OR NO CLEAR ORDER

TABLE 14H - PROBLEM DIFFICULTY TEXT D, GRADE 3

	L	S	O	E
25	-	-	-	-
50	13	S	O	C
75	18	S	O	X
100	16	S	O	C
125	18	S	H	X
150	20	L	O	C
175	15	L	O	C
200	15	S	O	C
225	13	S	O	C
250	10	S	O	C
275	-	-	-	-
TOTAL PAGES: 295	15.3			
RANGE: 10 - 20				

L - WORD LENGTH

S - NUMBER SIZE
S - 0-99
L - OVER 99
D - FRAC/DEC/PCT

O - O - ONE OPERATION
M - MORE THAN ONE
H - HEURISTIC NEEDED

E - C - NUMBERS AND OPERATIONS FLOW SMOOTHLY IN ORDER
X - MIXED ORDER OR NO CLEAR ORDER

TABLE 15 - PROBLEM SOLVING STRATEGIES TAUGHT IN 7TH GRADE TEXTS

	A	B	C	D	E
GUESS AND CHECK	2				6
DRAW A PICTURE	2	13			4
MAKE A TABLE	2				
MAKE AN ORGANIZED LIST	4				12
USE LOGICAL REASONING	20		2		11
WORK BACKWARDS	2				6
SOLVE A SIMPLER PROBLEM	2				7
FIND A PATTERN	12		2		
USE A FORMULA		24		33	32
MULTISTEP PROBLEMS		24			12
OPEN ENDED PROBLEMS					15
ESTIMATION WITH FRACTIONS					17
REASONABLE ANSWERS		25			7
CONSUMER APPLICATIONS					36
USING PROPORTIONS					12
SORT AND CLASSIFY					11
INVERSE OPERATIONS					14
USING GRAPHS		15			
USING A PERCENT FORMULA		15			
WRITE YOUR OWN QUESTION		13			
ANALYZING CONCLUSIONS		13			
USING PROBABILITY		16			
SAMPLING		14			
USING TABLES AND GRAPHS		12			
PROBLEM SOLVING WITHOUT NUMBERS				26	
PROBLEM SOLVING WITH SIMILAR FIGURES				7	
USING MONEY			24		
FINDING AND USING FACTS			9		
USING DECIMALS			14		
USING RECTANGLES			15		

TABLE 15 CONTINUED

PROBLEM SOLVING STRATEGIES TAUGHT IN 7TH GRADE TEXTS

	A	B	C	D	E
USING TRAPEZOIDS			10		
USING THE LCM			9		
CHOOSING THE OPERATION		41	21		9
USING INTEGERS			15		9
USING THE STRATEGIES	20				
FOUR STEP PLAN				88	9
READING INFORMATION					22
TOO LITTLE/TOO MUCH INFORMATION		12		10	8
ORGANIZING INFORMATION		43	16		21
USING ESTIMATION		25			21

APPENDIX I

LIST OF TEXTBOOK PUBLISHERS FOR SCOPE AND SEQUENCE INFORMATION Copyright 1985

A. Houghton-Mifflin, Mathematics
B. Merrill, Mathematics
C. Addison-Wesley, Mathematics
D. Harcourt, Brace, Jovanovich
E. Holt, Mathematics
F. Scott-Foresman, Invitation to Mathematics
G. Riverside, Mathematics
H. Harper and Row, Mathematics, 2nd Edition
I. McGraw Hill, Mathematics
J. McMillan, Mathematics

LIST OF TEXTBOOKS PUBLISHED FOR ALL OTHER INFORMATION

A. Addison-Wesley, Mathematics, 1987; Grades 3, 5, 7.
B. Harcourt, Brace, Jovanovich, Mathematics Today, 1984; Grades 3, 5, 7.
C. McGraw Hill, Mathematics, 1981; 5, 7.
D. Laidlaw, Using Mathematics, 1984; 3, 5, 7.
E. Houghton-Mifflin, Mathematics, 1987; 5, 7.

Discussion of Professor Clarkson's Paper

Nohda: Thank you, Sandra. The topics are problem solving instruction and the characteristics of word problems in elementary school mathematics textbook series in the U.S. Are there any questions or comments?

Clarkson: I have one comment. There are many different ways in which the textbooks can be examined. I want to know what other information would be useful in the final version of my paper. If you can think of some things that you would be interested in knowing about from these textbooks, just write them down and let me have the list. I will be happy to include that in my final paper.

Hashimoto: Thank you very much for your very useful information. You picked up 16 strategies, are they enough or not?

Clarkson: I think that if we teach a few strategies and teach them very well so students can use them, that is much more useful than trying to teach them 27 strategies simply because we can think of 27 names. I think that is what is happening in many of the textbooks. Whenever something can be called a strategy, a title is given and it is taught. But to use the strategies, the student needs to remember them. I think if we can suggest things like "Can you draw a diagram?" or "Can you think of a similar problem?" it can be helpful. If a student has a strategy which she/he has seen again and again, then it is more likely to be used. So, yes, I think 16 are enough if they are taught correctly. I think 8 are enough if we are teaching them so they are useful for the students. I think what we are not doing, however, is providing the students with enough problems where a strategy is a necessary part of the solution. So if we are teaching students strategies and then we give them problems that can be solved very easily without using that strategy, we don't really give them motivation to use those strategies.

Hashimoto: In the morning session results of the IEA study were presented. So I hope you consider the IEA results and analyze your data again. Strategies are related to the attitude questionnaire. This questionnaire includes how the teacher or student thinks about strategies. I remember one of the items, for example, checking an answer to a problem by going back over it. The IEA results are worth consideration.

Travers: I would make a comment and without looking up the data, my recollection is that with respect to that part of the questionnaire that was called "Mathematics in School," one of the things we asked teachers and students was how they felt about certain activities, how important they felt they were, whether they found them useful, and so on. And across the countries, students and teachers tended to think that things like checking their work were important but they didn't like to do it. That relates to my question, or maybe it is just a comment. To what extent is there any way to get a handle on the extent to which these very nice looking excursions into problem solving are in fact a part of the instructional program in the schools, or are they things that you do if you have time or if you have nothing else to do? Do the teachers regard these as part of the instructional program?

Clarkson: I am going to respond from the point of view of organization of the books. The textbooks, especially the newest ones, are organized so that the sections on problem solving are a part of the chapter organization. There are practice problems relating to what has been taught appearing later in sections that are "mixed practice" sections. There are all kinds of problems including problems where it might say, "Use guess and check to work this problem," or "Draw a diagram to help you solve this one." Those problems are interwoven into the practice section. The testing for the chapter, because most of the chapters are organized with a chapter test at the end, tests students on problem solving.

They are also being tested on problem solving in the unit tests. Problem solving is an integral part of the book. There are other sections that appear in chapters following the introduction of a strategy which involve applications of that problem solving strategy. In an organizational sense, the texts are organized so that the teacher will teach problem solving strategies.

Travers: For the information of people, if they want to follow up on the data to which Professor Hashimoto was referring, it is on pages 173 and 174 of my paper.

Whitman· Sandra, I wonder if you could tell us how the text handled evaluation of student learning of these various strategies. In other words, what kind of assessment was done or was provided to the teacher so that she could assess whether she in fact has made progress in teaching these skills? Were there any built-in evaluation materials for the teachers to use?

Clarkson: The built-in evaluation materials are the chapter tests and the end-of-the-unit tests in which the student is told to use a particular strategy to solve a problem. There were different kinds of materials available with each text that gave further practice. The teacher was cautioned to look at the procedures that the students went through and not just look at the answer. In many of the newest textbooks, there are group projects or class projects given that include gathering data and asking questions. These projects could involve developing a questionnaire or counting things that happened during a certain time period. I see many more group problem solving activities appearing in the newest textbooks.

Silver: I have a comment and then I have a question. The comment is that I think it was very good of Dr. Clarkson to mention the California State Mathematics Framework because I think some of the examples of the situational problems in mathematics and other subject areas and the group project notions that she just mentioned are directly related to the textbook publishers' desire to sell textbooks in the state of California. These areas are specifically called for in that document. The question I have for Dr. Clarkson is this: Did you see a corresponding change in the teachers' editions? For example, did you see change in the resource material available to the teachers, the suggestions to the teachers about how to teach problem solving, how to integrate the problem solving into the teaching of mathematics? How has that changed in the new editions of textbooks as compared with how things were before?

Clarkson: I don't have a large sample to choose from. I've seen that in some of the teachers' editions there are many suggestions about cooperative learning, working with groups, how to present problems to the class, how to lead discussions (especially in the problem situations that I showed you), questions that the teacher might ask, suggestions to read the problem through with the students and ask them to talk about the situation, and extensions like, "When you get through, what other things might you want to find out about this?" So there appears to be at least some more attention paid to the processes that the student will be using in solving problems and less emphasis placed on the answer that the students will get.

Shimada: One comment. Surely we can teach the procedure of problem solving objectively, but I doubt whether we can teach objectively the strategy of problem solving. Though I said I doubt the possibility of teaching it objectively I do not deny the possibility of teaching it subjectively. I have seen in many cases that a good teacher succeeded in teaching his own strategy to the students while other teachers who imitated him could not do so. This does not mean it is despairing to teach the strategy, but it means that the strategy is likely to link with the individuality of its owner, and differs with different people. In other words, for each one there may be a set of strategies suitable for his personality.

Wilson: I have a comment. I think it relates to this remark. The absolute worst experience I ever had in mathematics was someone who came and said, "I am going to teach this course as Polya would teach this course," and it was an absolute disaster, even though it was at Stanford. Let me go on and complete this because I think that one of the things that Dr. Clarkson has pointed out to us as a serious problem is that this business of four steps is so ingrained in school curricula nowadays that we are communicating to kids that if you just follow 4 steps, you can solve any problem. Anybody that has honestly solved a problem and reflected on that would know there is something seriously wrong with that belief. But, unfortunately, that is what is communicated to many teachers and to many students through that kind of setup. Just as an afterthought or a parenthetical remark on this, for heaven's sake, let's stop saying it's Polya's four steps. Polya never talked about 4 steps in any of his writings. He never demonstrated 4 steps to solving a problem in any of his teaching. Let's not blame him for this aberration of his view of problem solving.

Becker: I agree that the characteristics of the teacher play a very important role in generating student insight into solving problems. On the other hand, in my undergraduate elementary math methods course we spend considerable time on using various heuristics in problem solving. The way we approach it is to solve quite a large number of problems together, all the students and I. I pose the problems and we work together in solving the problems. Now, the problems are carefully selected so that when we have a solution, then we step back and we answer the question of what all was involved in going from the statement of the problem to the solution. Then we identify specific heuristics that were used and we continue to use those in solving other problems. So in that informal way I have demonstrated for myself that preservice teachers can learn these heuristics. But I think there has also been some formal research done, Schoenfeld is an example, in which it has been demonstrated that these heuristics can indeed be learned and applied by experimental subjects in problem solving.

Nohda: Thank you. Our time is up.

Professor Sugiyama's Paper

Kantowski: Professor Yoshishige Sugiyama is Professor of Mathematics Education at Tokyo Gakugei University. He has had classroom experience in the elementary school and also in the junior high school in Nagoya. He has spent all of his mathematics education career at Tokyo Gakugei University having been an Assistant Professor, Associate Professor and now a Professor. Professor Sugiyama is going to speak to us this afternoon on comparing word problems in Japanese and American textbooks.

Sugiyama: Thank you for your kind introduction.

A COMPARISON OF WORD PROBLEMS IN AMERICAN AND JAPANESE TEXTBOOKS

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Introduction

In this paper, Japanese textbooks are compared with American textbooks to make clear the different points of view in teaching problem solving and to get some suggestions for developing students' ability to solve problems. Textbooks titled "The Growth in Mathematics" (Center for Curriculum Development, Harcourt Brace Jovanovich, New York) are taken as the American textbooks and the "Atarashii Sansuu" (New Arithmetic, Tokyo Shoseki, Tokyo) as the Japanese textbooks. Before the comparison is made, some comments are given on the differences in the textbooks.

Textbooks are written so as to be useful in teaching-learning activities. If the aims of teaching and the teaching-learning activities differ, textbooks will differ. So, from the difference of textbooks, we can infer a difference in the teaching of problem solving. And at the same time, in order to compare word problems in the textbook of countries, it is necessary to recognize the difference in teaching-learning activities in those countries.

Japanese students learn mathematics as a group by hearing a teacher's explanation, or by answering the teacher's questions. This comes from the Japanese desire for uniformity. The Japanese do not like to differentiate among students. So, teachers feel compelled to teach all content in the textbooks to everybody. This is one reason to lessen the volume of content in the Japanese textbooks. The Japanese textbooks have fewer exercises than the American ones.

The American system takes individual differences into consideration. The textbooks are written so that students are able to learn according to their ability. American students, as is often noted, learn mathematics by reading textbooks and solving problems by themselves with occasional help from teachers. The American textbooks are written so as to allow students to learn by themselves. So, we can see more clearly what is to be learned on each page than in Japanese textbooks, which are written to be used with the help of teachers. The American textbook has many problems, but not all problems in the textbook need be solved. As a result, the American textbooks are larger and have more pages than Japanese textbooks.

Recognizing these differences and limitations, a comparison will be made. In the American textbooks we find problem-solving strands, named "problem solving help," which are intended to develop the problem solving ability. The strands in each grade are divided into 7-10 items. Each item has one theme which is known from its heading. All headings of "problem solving help" are listed at the end of this paper (See material at the end of this paper).

I would like to refer to the "problem solving help" headings to compare the textbooks. They show us what is to be taught there and they seem to suggest the American point of view for teaching of problem solving skills. But the Japanese textbooks do not have such headings. In the Japanese textbooks, problems are classified into "problem solving exercises," or strands titled "let's think about it." These headings suggest nothing about problem solving skills. Students cannot know what to learn there. That depends on the teacher's guidance. Different teachers give different guidance. So, the Japanese textbooks are not appropriate to refer to.

Exercises are not considered in this comparison of textbooks. Generally speaking, exercises in a unit are mere application of the content learned in the unit. After learning multiplication, exercises are given to apply multiplication; after learning division, exercises to be solved by applying division. Many of these exercises are easy one-step-problems. They cannot serve students in developing problem solving abilities.

Referring to "problem solving help" headings, the textbooks are compared from the following point of view; deciding the operation needed to solve problems; teaching of problem solving strategies; the degree of difficulties of problems; themes not found in the Japanese textbooks;

diagrams or pictures not found in the American textbooks; and the variety of problem materials.

1. On deciding the operation needed to solve problems

Some headings are related to deciding or choosing the operation needed to solve a problem as follows:

grade 3	
using subtraction	3(6-61)
add or subtract?	3(124-125)
add or multiply?	3(166-167)
add, subtract, or multiply?	3(188-189)
multiply or divide?	3(240-241)
problems without numbers	3(316-317)
grade 4	
add or subtract	4(40-41)
add or multiply?	4(152-153)
multiply or divide?	4(212-213)
grade 5	
do you have enough money?	5(88-89)
multiply or divide?	5(124-125)
add or multiply?	5(316-317)
grade 6	
which operation?	6(108-109)

There are many such pages, especially in the textbooks for lower grades. I think that this is desirable, because it is necessary for students to decide which operation is needed to solve each problem.

These headings are not found in the Japanese textbooks. Judging from this, the Japanese textbooks don't seem to consider them too important. If so, it might be a problem. But a Japanese teacher teaches the meaning of each operation at the same time he teaches computational skills. So, there is no need to especially teach it afterwards. I suppose that writers of the American textbooks plan to teach how to compute first, and application comes later. But in Japan, the meanings of operations are taught at the same time computational skills are taught. This indicates that Japanese put more emphasis on the meaning of operations. This will be clear when you see the textbooks for grades 1 and 2, in which students learn computational skills in addition and subtraction.

In the American textbook for grades 1 or 2, no word problems are found. There are only figures or pictures which suggest the meaning of addition and subtraction. The meaning of addition is the combining of two sets, and the increasing. The meaning of subtraction is the decreasing or the removing of a part from a whole. We cannot find the meaning of subtraction as the comparison of two sets.

Unlike the American textbooks, the Japanese textbooks for grades 1 or 2 have many word problems to be solved using addition and subtraction. The other meaning of subtraction, namely the comparison of two sets, is found in Japanese textbooks for grade 1, although the meaning of subtraction is mentioned in "problem solving help" for grade 3 in the American. I am of the opinion that it may be better to teach the meaning of an operation while computational skills are being learned. It will make students more likely to choose operations needed to solve a problem.

2. Problem solving strategies

Among the headings for "problem solving help," we find the following headings related to problem solving strategies. They are not found in the Japanese textbooks.

grade 3	
what's the question?	3(50-51)
make up an easier problem first	3(272-273)
grade 4	
using tables	4(60-61)
do an easier problem first	4(252-253)
making and using tables	4(280-281)
which answer is sensible?	4(298-299)
grade 5	
thinking about the remainder	5(150-151)
grade 6	
be sensible	6(230-231)

grade 7	
using equations to solve problems	7(40-41)
thinking about the remainder	7(102-103)
write a mini problem	7(124-125)
using formulas	7(230-231)
grade 8	
using equations to solve problems	8(40-41)
using equations to solve problems	8(60-61)
thinking about the remainder	8(108-109)
drawing pictures	8(300-301)

In the Japanese textbooks, problems only appear with little explanation about problem solving strategies. Occasionally, there are pictures or charts which suggest how to solve each problem. It is rare to give explanation about strategy for problem solving in the textbooks. But this does not mean that it is ignored.

The Japanese also wish to develop students' ability in problem solving. Since Japanese students learn mathematics under control of the teacher, problem solving strategies are taught by the teacher, not by the textbook. In the class of problem solving, teachers often ask questions such as, "What is given?", "What is unknown?", and so on. Through these questions students learn strategies for problem solving.

In Japan, traditionally, problems are classified into classes according to their structure or pattern, and students are taught the proper procedure to solve a problem included in each class. In the old days, students learned by heart the proper procedures to solve a problem. When a problem was given, students judged the pattern of the problem and applied the procedure fitted to it. That was the teaching method for problem solving in Japan.

This tendency might still remain a little among teachers or textbook writers in Japan. Occasionally, there are some teachers who teach only the proper procedure to solve a problem, and no strategy for problem solving. In such cases, it happens that a talented student may create strategies by himself, but many others can solve only problems like those they are taught. In that case, students think that mathematics, or mathematical problem solving is learning material by rote memory. This will not make students good problem solvers. Ability in problem solving does not consist of applying proper procedures to get the solution to a problem. At present, many teachers do not have such a point of view for problem solving.

But the Japanese must learn from the following headings seen in the American textbooks which are not found in Japanese textbooks, nor in classes of problem solving in Japan.

make up an easier problem first	3(272-273)
do an easier problem first	4(252-253)

These are very important strategies. In problem solving, there are cases where we must analyze a problem, divide it into several small problems, or make easier problems. After solving the small or easier problems, we can gain the solution to the original problem, or get suggestions for solving the problem.

3. Degree of difficulty of problems

Generally speaking, the degree of difficulty of problems in the Japanese textbooks are greater than in American textbooks. For example, let us look at "using equations to solve problems." The same content is taught in Japan too. But the level is higher than in the American texts. The problems in grade 7 or 8 in the American textbook are found in grade 5 in the Japanese texts. For example, the following problems are found in the Japanese textbook for grade 5.

(1) There is a water tank which is a rectangular solid with inside measurements of 25 cm. by 16 cm. and 20 cm. height. If 4 litres of water are poured into it, what is its depth?

Let x stand for the depth of water, and write an equation using x to express the volume of water in the tank.

Find out the value of x from the above equation. 5(79)

(2) 280 people were on a train. 58 people got off the train at a station and some people got on the train. There are now 307 people

on the train.

Let x stand for the number of people who got on the train and write an equation. Find out the number of people who got on the train at the station. (or, How many people got on the train at the station?) 5(79)

(3) I wish to buy some Japanese persimmons and a bag of chestnuts at the cost of 1500 yen. The price of a persimmon is 90 yen and the price of a bag of chestnuts is 780 yen. How many persimmons can I buy?

Let x stand for the number of persimmons and write an equation expressing the total cost.

Find out the value of x from the above equation.

How many persimmons can I buy? 5(81)

(4) The total cost of a half dozen pencils and an eraser is 350 yen. The price of an eraser is 80 yen.

Let x stand for the price of a pencil and write an equation and find out the cost of a pencil. 5(81)

The high level of problems in the Japanese textbooks could be exemplified by other examples, say, by problems which are solved using more than one step. The following headings are found in the American textbooks from grade 6.

more than one step	6(166-167)
more than one step	7(252-253)
more than one step	8(236-237)

On the other hand, there are problems which are solved by more than one step in grade 2 or 3 in the Japanese textbooks. Some examples follow:

(5) 1 dozen pencils is the same as 12 pencils. I bought 2 dozen pencils and used 8 pencils. How many pencils remain? 2b(17)

(6) Yoshiko was given 300 yen by her father and 250 yen by her mother. She spent 470 yen to buy a book. How much does she have now? 2b(55)

(7) There are 5 bottles with 18 dl. oil in each. If you divide the oil into 3 bottles, how much oil is in each? 3b(16)

The foregoing examples and others show that the degree of difficulty of problems in the Japanese textbook is higher than in the American.

4. On the themes not found in the Japanese textbook

The following headings, which are concerned with collecting and selecting information, are found in the American textbooks, but not found in the Japanese textbooks:

grade 3	
too much information	3(300-301)
grade 4	
too much information	4(172-173)
grade 5	
too much information	5(36-37)
missing information	5(216-217)
grade 6	
missing information	6(316-317)
grade 7	
more than enough information	7(302-303)

In the American textbooks, we find these activities in the lower grades. But these are seldom found in classes of problem solving in Japan.

Nor are the following found in Japanese textbooks:

grade 3	
information on maps	3(213)
grade 4	
reading maps	4(124-125)
grade 6	
be an alert shopper	6(206-207)
grade 7	
being a wise shopper	7(142-143)
reading maps	7(160-161)
grade 8	
reading maps	8(86-87)

using recipes	8(124-125)
being a wise shopper	8(144-145)

They also are related to collecting or selecting information. They show a difference in educational expectation in problem solving between the American and Japanese textbooks.

I think that Americans recognize the importance of collecting and selecting information in problem solving. In real problem solving situations, we must find necessary information from the real world. Americans think much of solving real world problems.

The problems in a textbook are models of problems which occur in the real world. In making models, we must select information, neglect some of it, and make some assumptions. These must be more important activities in the computer age. Japanese educators must reconsider this.

5. Diagrams, or pictures, helpful in solving problems

One thing found in the Japanese but not in the American text is the diagram, or picture, helpful in solving problems. Among the "problem solving help" there is an item titled "drawing pictures." But the picture is a mere sketch of the problem situation. It is not the same as the Japanese. In the Japanese textbook, diagrams, pictures or illustrations are given in the form of a band, segment, line of numbers and so on, which point out the structure of the problem and are helpful in solving the problem.

Drawing these diagrams or pictures has two purposes. First, the teacher explains the problem solving procedure using it. Second, students utilize it to solve a problem. The diagram or picture points out the relation of elements in the problem and helps students to decide the operations needed to solve it. It is especially helpful in solving complicated problems or problems to be solved using inverse operations. It would be a useful skill for problem solving. It is not very easy to draw the diagrams, but it is helpful in solving problems. Students who have enough experience to see and to draw them will be good problem solvers.

6. Variety of materials

Although it is not directly concerned with problem solving ability, I would like to refer to problem situations or materials, because they show us the difference in cultural and educational points of view. The textwriters in both countries take students' interests into consideration. So, they select situations or affairs from the environment of the students. But American textbooks have a freer sense than the Japanese. Japanese also select situations or affairs near to students, but these situations are felt to be fictitious. In the American textbook, there are more kinds of cake or foods, more types of sports and other things than in the Japanese text. And we cannot find in the Japanese textbooks such things as the following:

> The number of votes which U.S. Grant gained in becoming president in 1872.
>
> A flea is 3 millimeters long. It can jump 100 times its length.
>
> A caterpillar on a tree is 4 centimeters long. The height of the tree is 200 times the length of the caterpillar.
>
> Some pirates bury 6 treasure chests. In each chest there are 27 pieces of jewelry.
>
> Peter ate 4 pumpkin pies. He went to the store and ate 2 more. Now Peter's stomach aches.

It seems that the Japanese have a limitation in mind, intentionally or tacitly, when selecting situations or materials.

Simply speaking, Japanese do not use such things as follows; creepy insects or animals such as a flea or a caterpillar and so on; inadvisable acts such as overeating which causes stomachaches; political affairs; and bad people like pirates. In addition, though shopping is taken up in the Japanese textbooks, trades or businesses in which making a profit or losing money come into question are not taken up. Stories of students working in shops or factories are found in the American textbooks, but not found in the Japanese textbooks. Japanese teachers do not talk about the hard times that adults sometimes have. Similarly, Japanese do not wish to talk about misfortune (for example, death or illness) or unhappiness in teaching. The Japanese textbooks do not take up immoral or unhappy situations. Instead of saying "he died at age 80," the Japanese textbook says "he lived till 80." But the American textbooks take up freely anything in which students have an interest.

What influence do such limitations have on developing ability in problem solving or learning mathematics? In the first place, it affects the interest of students in mathematics and problem solving. Students will feel familiar with mathematics when situations are interesting for them. In the second place, if students feel situations to be fictitious as in the Japanese textbooks, they will think that mathematics has nothing to do with them. If the situations are real, students feel that mathematics is important for them, and mathematics deserves learning.

7. Conclusion

The Japanese textbooks are compared with the American textbooks and the following conclusions are reached:

(1) The meaning of the operation and the appropriate operations needed to solve a problem is considered much more in the Japanese textbooks than the American texts.

(2) Problem solving strategies are found in the American textbooks. These are not found in the Japanese textbooks, but they are taught by teachers. Proper procedures in solving a problem, not strategies, are taught in Japan.

(3) The level of difficulty of problems in the Japanese textbooks is greater than in the American textbooks.

(4) The American textbooks recognize the importance of collecting and selecting information. These things are not found in Japanese textbooks.

(5) Diagrams or pictures helpful in solving a problem are found in the Japanese textbooks, but they are not included in the American texts.

(6) The variety of materials for problem solving is richer in the American textbooks than in the Japanese texts.

The above mentioned are some of the differences between the American and Japanese textbooks. In a sense, they show the difference of teaching of problem solving, too. From these, some implications for the teaching of problem solving are suggested, as follows. The basic difference seems to lie in the expectations concerning teaching problem solving, or in the nature of problems to be solved. The Japanese expect students to be able to solve difficult problems. The Japanese think much of solving problems of a mathematical nature and difficult ones, while Americans think much of solving problems in daily or real life. Situations in daily life are found in the Japanese textbooks, but they are not really related to daily life situations. The Japanese seem to expect mathematics education to develop the thinking faculty and to develop problem solving ability through learning mathematics.

The American expects students to develop ability to use mathematical skills for solving problems in daily life. Learning equations is not only an aim for itself, but also is to be utilized to solve daily life problems. Thus, problems in the American textbooks are not complicated ones. In contrast, the Japanese expect mathematics teaching to develop thinking ability, but prefer complicated and difficult problems. Americans, who expect to develop the ability to solve daily life problems, are not satisfied with solving given problems, but they think much of selecting information or formation of problems. Selecting information and formation of problems will be important activities in the computer era, or in the information age. Japan, I think, must learn this from America.

One problem remains: Which is to be more emphasized, mathematics teaching to develop thinking ability, or to develop the ability to solve problems in daily life?

Materials

The titles of strands in American textbooks (in the order of grade)

Grade 3

what's the question?	3(50-51)
using subtraction	3(60-61)
how much?	3(72-73)
do you have enough money?	3(98-99)
add or subtract?	3(124-125)

add or multiply?	3(166-167)
add, subtract, or multiply?	3(188-189)
information on maps	3(213)
multiply or divide?	3(240-241)
make up an easier problem first	3(272-273)
too much information	3(300-301)
problems without numbers	3(316-317)
Grade 4	
add or subtract	4(40-41)
using tables	4(60-61)
do you have enough money?	4(86-87)
making change	4(110-111)
reading maps	4(124-125)
add or multiply?	4(152-153)
too much information	4(172-173)
multiply or divide?	4(212-213)
dividing with money	4(234-235)
do an easier problem first	4(252-253)
making and using tables	4(280-281)
which answer is sensible?	4(298-299)
Grade 5	
too much information	5(36-37)
making change	5(60-61)
do you have enough money?	5(88-89)
multiply or divide?	5(124-125)
thinking about the remainder	5(150-151)
missing information	5(216-217)
more than one step	5(300-301)
add or multiply?	5(316-317)
Grade 6	
do you have enough money?	6(76-77)
which operation?	6(108-109)
more than one step	6(166-167)

be an alert shopper	6(206-207)
be sensible	6(230-231)
area or perimeter?	6(296-297)
missing information	6(316-317)

Grade 7

estimating costs	7(14-15)
using equations to solve problems	7(40-41)
which whole number is the answer?	7(60-61)
thinking about the remainder	7(102-103)
write a mini problem	7(124-125)
being a wise shopper	7(142-143)
reading maps	7(160-161)
using formulas	7(230-231)
more than one step	7(252-253)
more than enough information	7(302-303)
pocket calculators	7(316-317)

Grade 8

using equations to solve problems	8(40-41)
using equations to solve problems	8(60-61)
reading maps	8(86-87)
thinking about the remainder	8(108-109)
using recipes	8(124-125)
being a wise shopper	8(144-145)
what score do you need?	8(210-211)
more than one step	8(236-237)
using square roots	8(252-253)
drawing pictures	8(300-301)
pocket calculators	8(316-317)

Discussion of Professor Sugiyama's Paper

Kantowski: Thank you, Professor Sugiyama. Your English was very good in your lecture. We also appreciate the examples which help us to understand many of the differences between your textbooks and ours. I would like to open up the discussion now.

Clarkson: I would like to make a comment. I also want to tell you that your English is very good. I want to say that I am very pleased that you and I spoke about things that are so similar. I have learned a great deal from your presentation and I think that it was very good for us to have these two so close together on the program. Thank you very much.

Silver: I have a question since we discussed earlier the influence that textbook publishers in the United States have on curriculum. I am trying to understand how textbooks are selected in Japanese schools. Is there a choice, and if there is a choice, how is the choice made? Is it made at a teacher level, at a school level, or at a prefecture level? How exactly does the selection of textbooks occur in Japan?

Sugiyama: At the elementary and junior high school levels, textbook selection is made by smaller units than the prefecture, rather by districts; whereas when you get into the senior high school, then each school has the option as to what text to use.

Hashimoto: In addition to this, elementary and junior high school textbooks about mathematics are published from six companies now. This means six different books.

Miwa: All textbooks are compiled by commercial publishers, but they depend upon the Course of Study which is established by the Ministry of Education. Of course, the details are left to the publishers. So there are six kinds of textbooks in elementary and junior high school in Japan; however, the subject matter of all textbooks is the same but the methods to develop the content and expected teaching methods may vary from publisher to publisher.

Wilson: They are produced by commercial companies, but are there professionals such as math education people or mathematicians writing? Are they involved in the writing of textbooks?

Shimada: Both mathematicians and mathematics educators as well as teachers are involved in writing textbooks. Usually they make up a team and have a responsibility for the content, while editing matters such as layout and illustrations are left to specialists in the publishing company.

The Ministry of Education checks all of the content from page one to the last page. I had been an official in charge of authorization in the Ministry. Criteria of authorization (checking) are: whether the objectives and scope of the content are in accordance with those in the Course of Study; whether the content is suitably arranged for realizing the objectives; whether sentences are readable for students' level, etc. Most of the examiner's time is occupied by checking all of the exercise problems.

After the Ministry gives an "okay" to publishers, then books are exhibited in various parts of Japan for inspection by those concerned with the selection of textbooks. They decide one from six in the case of elementary and junior secondary school.

Variation between different publishers is usually on its method or approach to the mathematical content, not in the content itself.

Wilson: How often are they revised?

Shimada: They are revised usually every 3 or 4 years. Revision is made based on criticisms from users--teachers. It is an interesting fact that though there are many characteristic features in each of the first versions of textbook series, when the Course of Study is revised by the Ministry of Education, they become similar and lose some of their characteristics after several revisions. This is caused by critiques which say that such and such activities are not found in this series, but others include them.

Wilson: There are six because there are only six approved by the Ministry?

Shimada: Yes, this is the case for elementary and junior secondary schools, i.e., the compulsory schooling level. At the senior secondary school level, there are more than 20 series, but I do not know the exact number.

Wilson: At the upper secondary level, can any publisher who wants to get into that issue a book? Can they be an entrepreneur at the upper secondary level, whereas at the elementary it's more restricted?

Shimada: Editing and publishing textbooks cost much money, and it requires for publishers to wait for at least three years between completing the draft version for the Ministry's checking and getting any actual income. And much more money is needed for publishing textbooks for the compulsory schooling level than for senior secondary school for several reasons.

Wilson: This is very similar to our situation. We have more than 6 publishers in it but the marketplace determines that, and if you take the first six in terms of their sales, it's over 90% of the market. I am on a publication team. There are 16 math education people who are on this elementary series team. But there is also another corresponding group of people working for the company as editors and they make many of the decisions that deal with content and sequence. There are team meetings and alot of interchange, but the publisher makes many of the decisions of what the product looks like and, in part, that draws very heavily from the response in the field on the tryouts and what the salespersons find that the teachers are looking for - as Dr. Clarkson said, what will sell has a strong impact on the structure of the materials. We estimate, different companies vary a little bit, but they estimate in the range of a $5 million investment over a 5-year period of time as a minimum to have a textbook series for elementary school ready. And that is before any sales, so that's an investment that a company must make and that's a minimum. Some of them invest more.

Whitman: I just want to make a comment. On one of the books that was given to the American delegation, the large one, there's a very nice diagram showing the process of textbook selection. So those of you who are especially interested in that might take a look at that flow of processes.

Wilson: I have one other question that I would like to raise that maybe we don't want to answer now. In light of the comment that was made here earlier that the textbooks differ because the teachers are expected to do different things, what I would like to know is something about the kind of training provided to elementary teachers, to be prepared to use a textbook that has very little explanation for certain problems. There must be differences for two companies. Maybe this isn't the right time to have that discussion, but I'd like to see it brought up.

Nohda: Ten or twenty years ago in Japan, at first each of the companies was producing a text particular to them. And it was a characteristic of textbook production that the Ministry of Education had certain preferences, as well as the publishers each having their own preferences in textbook production. About ten years ago textbooks were free; prior to that, each student in the public school had to purchase them. Since they became free, the page numbers became much less. There is a lot less area within which to maneuver in a slimmer volume. And textbook publishers and writers of textbooks do comparative studies with texts published in other nations, much as the United States and other places.

Sugiyama: As far as the earlier question about whether there is any inservice training for teachers on how to use or how best to use a particular text, one can say that there is no such training program. But then again, in Japanese schools it is common practice for teachers to visit other classrooms, to observe the teaching of other teachers and there is much going back and forth on ideas as to the best ways to teach, more interaction among the teachers in dealing with teaching style. And since the textbook publishers and the editors like to reflect the voice or the different voices or individual styles of teaching among the various teachers, there is an effort made to create texts that include not just one way, one method or one style, but those that reflect many teachers' voices and many styles so that any teacher can use them effectively. One can pick the voice he likes or the style that he likes best.

Becker: Professor Sugiyama, you mentioned that the meanings of operations are taught at the same time computational skills are taught. I wonder if you or some of the other members of the Japanese delegation could talk a little bit more about that. For example, are concrete objects used by children in learning the concepts?

Sugiyama: Concrete objects are used. In the case of 3 + 2, the children actually handle the blocks. The meaning of operations, combining 3 with 2, is established by concrete objects. In another example, where objects are added to existing objects, the children themselves handle them in bringing these sets of objects together.

Becker: In the United States, sometimes primary teachers use Diene's blocks or multibase blocks which can be useful in getting an understanding of place value and, hopefully, to understand the addition-subtraction algorithms and so on. Do you use these materials or similar materials?

Sugiyama: They are not used in Japan. Teachers know about them but they don't use them.

Becker: Or, I'm wondering if there is a variation of those they use that's based on 5 and 10?

Sugiyama: It's not used.

Becker: Whenever I see the blocks used they are grouped by 5, is that correct?

Sugiyama: They are only marks for five.

Wilson: I was just going to comment that I think the growth in mathematics text may have indicated or conveyed that we don't pay attention to meaning of operation at the time we are introducing the concept or the algorithm, but I think most textbooks do have that these days and make a point of that. And just going back to my previous question, my question about teacher education is not an inservice question, it is preparation, pre-service. How do we get there and I just have a sense you are doing something different from what we are.

Sugiyama: I don't know whether the Japanese system differs from that of the U.S., but in the training of teachers, there is a course in educational materials and also some actual teaching in lab schools, or they go out to public schools. Student teaching varies from university to university from two weeks to six weeks duration.

Wilson: How much mathematics would an elementary teacher have studied?

Sugiyama: It varies from university to university, but ordinarily four credits (one credit is an hour per week for thirty weeks).

Wilson: I do get the impression that your teachers study more mathematics than our teachers might.

Miwa: To become an elementary school teacher, mathematics is not compulsory, but is optional. But in the university or college, students are required to get 36 credits of general education, and usually mathematics is one of those subjects. Many students select mathematics.

Shimada: I have studied the legal requirement of mathematical training for elementary teachers. The minimum requirement is no training in college or university. The second level is 2 hours per week for one semester, which is for either one of mathematics as a part of general education program or mathematics training in the courses called 'subject studies' in the professional training programs for teachers (study in arithmetic materials). The third level is both of above, and the fourth level is the above together with some more academic mathematics courses.

If a student wants to be qualified as an elementary school teacher, especially with much emphasis on music or art, he or she may not select any mathematics; at least it is possible. But usually students who want to be common elementary school teachers will select at least the course 'study in arithmetic materials,' because if they cannot teach arithmetic they will be regarded as a poor classroom teacher who is responsible for all aspects of children's education. They also take a course in the national language for the same reason.

There are two kinds of ordinary teacher licenses in Japan, which are called the first class and the second class. Some students who are preparing for the first class license of lower secondary school also prepare for the second class license of elementary school, as additional training needed for the former. Those who select music, art, or some others which need training for specific skills as their first choice of specialty for the first class license of lower secondary school teacher may get their job in a large elementary school. In such a case, it is a common practice that they serve like a specialist teacher (in the old system) who is in charge of teaching only their specialized subject to many different classes, while being free from the work of general classroom teacher. Most of elementary school teachers having no mathematics training are found among them. Those who select academic subjects as their first choice for the first class license of lower secondary school teachers usually select at least the second level of mathematics training as they expect to be elementary teachers if they cannot get a job in the secondary schools.

Silver: I'll ask a two-part question so I can get it all in. One part has an inservice flavor; is there a teacher's manual, an instructional guide, which comes with that thin textbook for first or second grade? The second part of my question is related to the slides you showed and the videotape we saw yesterday. Both of the elementary teachers depicted were male, but in the United States the predominant sex of elementary teachers is female. Is that different in Japan or are these just aberrations?

Sugiyama: For the first question, yes. There is a teacher's manual.

Silver: Is it about as thin?

Sugiyama: Double that. For the second question, there are about 45% male teachers in the elementary schools.

Silver: What would the percentage be at the lower and upper secondary levels?

Sugiyama: At the middle school level, lower secondary, it's about 70% male for mathematics teachers. Even more than that at the high school level--over 90%.

Wilson: Let me see if I understand a statement that was made here a moment ago. You said that teaching of the National Language and the teaching of mathematics were high status. That implies there is an implicit stratification status built-in to being the mathematics teacher in the elementary school. Is that correct? That surely would be a difference between Japan and U.S. school classrooms.

Shimada: No. There is no stratification status built-in.

Sugiyama: But there are those who by choice prefer to teach only the lower grades, whereas there are those who by choice teach only upper grades. One can draw one's own conclusion as to which is higher or which is lower in terms of status.

Travers: I would like to pursue the issue of teacher education in the following sense. It has to do with the culture of the school and the culture of the university. We heard an allusion yesterday, Mr. Hashimoto saying it was his perception that there seemed to be a lot of separation or isolation between the school and the university. In the American context, at least, I think he was referring primarily to Illinois, as compared with the Japanese. This phenomenon in the U.S. manifests itself in several ways and one is very pernicious. Let's say, for example, that we have a teacher education program that does a lot in problem solving and we have these marvelous strategies worked out and the student teachers have worked through very good books on teaching mathematics and so on, then they go into the schools and the culture of the school says, "You forget now what you have been doing because you are in the real world of teaching and you've got a syllabus to follow and all of these classes to meet," and so on. So there is really a separation between the culture of the school and the university, at least in Illinois, and I am wondering whether that is perceived as a problem in Japan?

Whitman: I just want to make a comment and then raise a question. One is somewhat along the line of what Ken was speaking about. One of the things that to me seems to make a difference between the way Japanese and American elementary teachers function is that in Japan the teachers don't stay at one grade level. They move and I think that they get a broader view of the total curriculum so that they might have a better idea of what might be significant. They move year to year. I think that, if I were conjecturing, that makes a lot of difference in their ability to handle the classes in a more effective and efficient manner. I have a question to Professor Sugiyama. In my visits in the lower middle schools, I noticed that all the teachers had students purchase a workbook or homework book. I was curious about the elementary school, do elementary school children have to buy that also? Do those books address problem solving? That's my next related question.

Sugiyama: Yes, almost all elementary school children have workbooks which are used for exercises of algorithmic procedures. In Japan, we find many sorts of such books, and some address problem solving. In addressing Professor Travers' first question, I have some comments. In Japan one gets the impression that the new teachers work hard at learning from established teachers at the school as well as attending workshops and meetings that might in many cases include university colleges of education faculty, and that they seem to put in more hours attending meetings and workshops than perhaps in the United States.

Travers: That comment ties in very nicely with another piece of IEA data that had to do with frequency of teachers' meetings in the school. The frequencies were very comparable in Japan and the United States, I think about once or so a month, but the content was rather different. The content of the meetings reported by the two countries was that in the United States they tend to be matters dealing much more with procedural kinds of things such as filling out forms, attendance, and what we might call our "paperwork" sorts of activities. The Japanese teachers reported the meetings as having much more to do with the content of what they teach, strategies, how they handle mathematics, and more of what we would call professional kinds of concerns.

Kantowski: I'm sorry, but we are nearly out of time. Dr. Becker?

Becker: There was one little question that I had which follows up on what Nancy was asking. I am asking the Japanese now, how often does it occur that a teacher with a group of students in first grade goes to second grade with the same group and to the third grade with the same group, and so on?

Sugiyama: It's usually done in units of two years and usually does not go beyond that. A first grade teacher might follow her class into the second, but most likely not into the third year.

Nohda: Two comments. There are cases where the first and second grade teacher might take the same class all the way up to the third grade, but after the second grade the students are divided evenly and teacher does not have an entire class of same students following them up.

Next comment, the beginning or new teacher in the Japanese elementary schools usually begins his/her career at the third or fourth grade levels. They are not given a first assignment for either the first/second or fifth/sixth grade levels.

Wilson: I have a couple of comments. One is that a year ago I was in a similar joint seminar between U.S. and professionals from Italy. In Italy it is educational policy that the teacher will start with the first grade, their first year and follow that same group of students through four years or five, whichever is in their elementary level. They feel very strongly about the same teacher staying with the same group of students. There is very little mobility professionally so that it is a system that works. It is an interesting contrast.

Some of us in the U.S. delegation were talking and, for the benefit of the Japanese colleagues, I feel we ought to elaborate a little bit on the textbook selection situation where the guidelines in California were mentioned as dictating the structure of textbooks. The reason for that is that our textbooks tend to be adopted on cycles of about five years. The California adoption is coming up in 1987 and California has a set of guidelines. It is a state in which a publisher must be on the approved list to sell its textbooks. So it is very important to every textbook company to be on that list. That's the difference between success and non-success. First of all, California has a very large population, so it happens that way; second, California and Texas are the two states that have the state approved list with large enough populations to force textbook companies to cater directly to their wishes. In effect, California dictates what will be in the textbooks that are published in 1987 and for the next five years, so the rest of us will use those textbooks. It's K-12 in Texas, but there is not nearly as much of an impact at the secondary level because there are many more players in the game and you don't necessarily have whole series being adopted.

Travis: I was going to mention earlier about the effect that curriculum can have on textbooks and I was wondering if there was any such driving force that affects the curriculum in Japan? Or is there just a nationally mandated curriculum set by the Ministry? Is there a committee that helps determine that curriculum and that, say, every seventh grade student will study the same set of topics, and so forth?

Sugiyama: It is a fact that the Ministry of Education determines the curriculum, but the curriculum is really set by a group of advisors selected from various places in Japan. You are right in assuming that in Japan a seventh grader can move to any prefecture and receive or study under the same guidelines of a national curriculum. That's why this refers back to what Professor Shimada was saying earlier about a group selected to study texts, offer suggestions and standardize them in that way. '

Silver: I have one comment, as an amendment or addition to Dr. Wilson's comment, because he made the process sound both a little undemocratic and uncapitalistic. Actually once an adoption list exists, a textbook publisher that is not on the list can still sell textbooks in the state of California, but the school districts cannot use state funds which are specifically made available for the purchase of textbooks. Therefore, those school districts would have to use all of their own money to do that and that's very impractical; so the adoption list really doesn't cut off free trade, it just makes it practically impossible to sell textbooks.

Kantowski: Thank you all for a very lively discussion. Now we have to adjourn.

Professor Ishida's Paper

Silver: It is a pleasure to chair this morning's session and to introduce this morning's speaker, Professor Tadao Ishida, Associate Professor of Mathematics Education at Hiroshima University. Professor Ishida has been at Hiroshima University for several years after having served as an Assistant and Associate Professor at the University of Yamaguchi. Professor Ishida is a veteran of many international meetings and attended the Fourth International Congress on Mathematics Education in Berkeley and the Fifth International Congress in Adelaide, Australia. He also attended the Eighth International Meeting of the International Group For Psychology of Mathematics Education in Sydney, Australia. It is my pleasure to introduce Professor Ishida who will speak on issues of language factors in performance of Japanese students on word problem solving.

ISSUES OF LANGUAGE FACTORS AND PERFORMANCE OF JAPANESE PUPILS IN WORD PROBLEM SOLVING

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1. Purpose and Method

On the subject of language factors which exercise influence over school children in their word problem solving, there are some themes that should be examined from various angles that apply to many kinds of word problems. In this report, it is the researcher's aim to focus on language factors that exercise influence on school children in their solving of word problems which include one addition or one subtraction process.

Carpenter (1982), Nesher (1982), Riley (1983), et al., classified the aforementioned word problem into the four categories "Change Problems," "Combine Problems," "Compare Problems," and "Equalize Problems" and studied the children's process of problem solving and their relative difficulty. In this report also, the researcher tried to show some of the "findings" on these issues as itemized below, after analysing the results of the study which was done with second and third grade children. He gave them 14 different types of word problems which belong to Change, Combine, and Compare problems, and there are some with different wording of the same problem types. Comparisons were also made of the results of performance between the children of Japan and the U.S.:

(1) Effects that words and expressions exerted upon school children in solving word problems.

(2) A comparison between Japanese and American school children in their performance on each type of problem.

(3) A test on the hypothesis of Relative Difficulty in word problems, which was suggested by P. Nesher, J.G. Greeno and M.S. Riley.

2. Problems tested and their results

Examples of tested problems are as follows:

Change (1a)--Result unknown

Takashi had 5 caramels.
Then Masao gave him 3 caramels.
How many caramels does Takeshi have now?

$$\boxed{5} \xrightarrow{(+3)} \boxed{?}$$

Change (1b)--Result unknown

Akiko had 5 marbles.
Then she was given 3 marbles by Satomi.
How many marbles does Akiko have now?

Change (1c)--result unknown

There were 5 swallows.
4 swallows came flying.
How many swallows are there now?

Change (3)--change unknown

Takeshi had 3 caramels.
Then Masao gave some caramels to Takeshi.
Now Takeshi has 8 caramels.
How many caramels did Masao give to Takeshi?

$$\boxed{3} \xrightarrow{(+?)} \boxed{8}$$

Combine (2a)--Subset unknown

Takeshi and Masao have 8 marbles altogether.
Takeshi has 3 marbles.
How many marbles does Masao have?

Compare (3)--Compared quality unknown

Akiko has 3 marbles.
Satomi has 5 more marbles than Akiko.
How many marbles does Satomi have?

The full set of tested problems are shown on Appendix I and their results are as follows:

Table I

Percentage of Success of 14 Types of Word Problems in Japan and the US

		2nd Grade		3rd Grade	
Problem Type	Formula	JAPAN (n=38)	US*	JAPAN (n=34)	US*
Change (1a)	5 + 3 = x	63	100	74	100
Change (1b)	5 + 3 = x	97	---	100	---
Change (1c)	5 + 4 = x	58	---	82	---
Change (2)	8 - 5 = x	97	100	91	100
Change (3)	3 + x = 8	76	100	85	100
Change (4a)	8 - x = 3	92	100	100	100
Change (4b)	8 - x = 3	92	---	97	---
Change (5)	x + 5 = 8	82	80	77	95
Change (6)	x - 5 = 3	76	70	85	80
Combine (1)	3 + 5 = x	100	100	100	100
Combine (2a)	3 + x = 8	92	70	88	100
Combine (2b)	3 + x = 8	97	---	97	---
Compare (1a)	8 - 5 = x	87	85	91	100
Compare (1b)	8 - 5 = x	100	---	100	---
Compare (2)	8 - 5 = x	100	75	100	100
Compare (3)	3 + 5 = x	87	80	94	100
Compare (4)	8 - 5 = x	97	90	100	95
Compare (5)	8 = x + 5	47	65	79	75
Compare (6a)	3 = x - 5	29	35	68	75
Compare (6b)	3 = x - 5	42	---	77	---

*Manipulative objects were provided. Source: Riley (1983)

3. Analysis and Discussion

Effects of words and expressions

1.1. When comparing the percentage of correct answers of the same problem types, Combine (2a) and (2b), it was found that the percentage is a bit higher in the case of (2b) as shown in Table I. The reason for this could be attributable to the fact that in (2b) case, they used the key word "the rest" which implies subtraction. This fact coincides with the findings reported by Carpenter, et al., (1981). On the other hand, in case of Change (1a) and (1b), also the same problem types, the result showed (1b) considerably higher than (1a). This may possibly be attributable to the fact that in the (1a) case there is the word "gave" which implies subtraction whereas in (1b) case, there is the phrase "was given" which implies addition.

The above fact and the other survey results lead us to the following:

1.1.1. Whenever there is a word or phrase which implies addition such as "was given," "increased," or "came" in the problems of addition, or in subtraction problems, words such as "gave," "decreased," or "ate" are used, the percentage of correct answers increases in comparison to other problems where these words and phrases are not used.

1.1.2. Where there are words which could imply subtraction in the problems of addition, or words or phrases which could imply addition in the problems of subtraction, the percentage of correct answers decreases.

1.2. To compare Change (1c) with (1b), the percentage of correct answers for Change (1c) is considerably lower as shown on Figure 1.

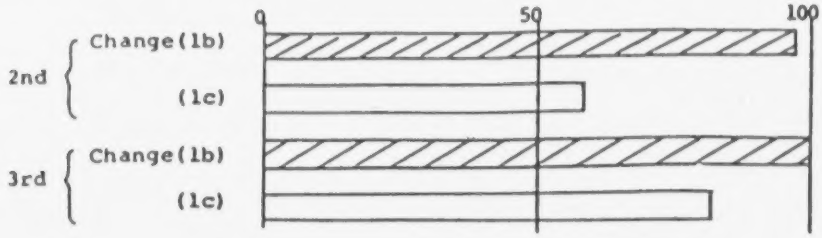

Figure 1

Most of the wrong answers in (1c) were expressed as 5 - 4 = 1 (5 swallows minus 4 swallows equal 1 swallow). The reason would be that the Japanese expression "Yon wa tonde kimashita," (4 swallows came flying) might be confused with "Yon wa tonde ikimashita" (4 swallows flew away). Apparently, the expression that could imply subtraction along with wording and phonetical similarity confused the children. Thus, they answered in subtraction. In this way, where there are similar expressions both for addition and subtraction, the children are likely to be confused because of them, and accordingly the percentage of correct answers would be lower.

1.3. The correct answer percentages differ also between Compare (1a) and (1b). The reason for this could be that in the Japanese expression in (1b), the key word "difference" could imply subtraction.

1.4. Change (4b) and Compare (6b) are word problems made out in a little simpler forms of expression of Change (4a) and Compare (6a), respectively. As the supplement of Figure 2 shows, the difference of correct answer percentages between (a)-type problem and (b)-type problem is the greatest in Compare (6) for 2nd graders.

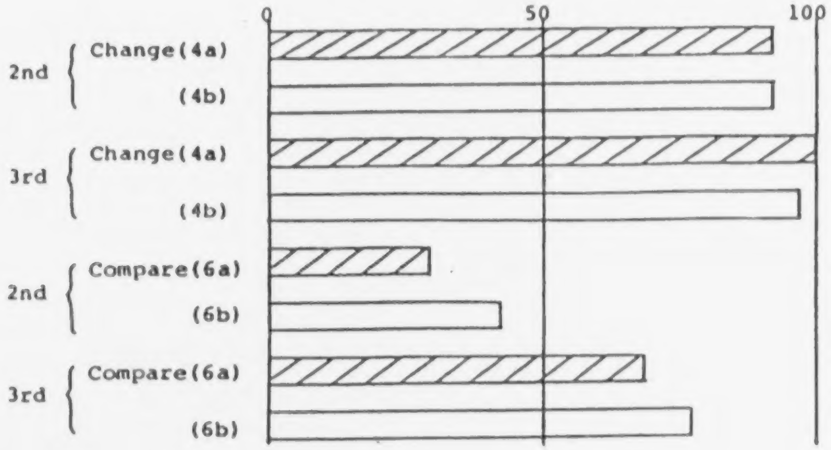

Figure 2

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As we see later in item (3), Compare (6) is one of the most difficult word problems among the 14 types that were the object of this research. The age group of second graders, generally, is the age when they just start to realize the structure of problems. From these considerations, the following could be suggested.

1.4.1. Until the children's ability advances to the stage where they can realize the problem structure, a simpler form of expression could help them to solve the difficult type of word problems easier.

1.4.2. When the children's ability advances further to a more mature stage and they can fully realize the problem structure, the form of expression does not make much difference in the children's ability to solve those word problems, regardless of the complexity of the expression.

(2) Comparison of performance between Japanese children and their American counterpart

The survey results shown in Table I indicate the result of "the manipulative object" being provided to the American children, whereas their Japanese counterparts were not subject to it. Therefore, a fair comparison simply cannot be made in this instance. The survey result reported by Nesher (1982), as shown in Table II, is the one with no use of "manipulative objects" in the word problems for American children, and of which show a considerable number of incorrect responses.

Table II

Type of Problem	Percentage of Success (2nd-6th)
Change (1)	82
Change (2)	75
Change (3)	72
Change (4)	77
Change (5)	48
Change (6)	49
Combine (1)	79
Combine (2)	52
Compare (1)	76
Compare (2)	66
Compare (3)	65
Compare (4)	66
Compare (5)	60
Compare (6)	54

Taking these results into consideration, the researcher made a comparison between Japanese and American children in their correct performance percentage on the basis of Table I as described below:

2.1. Generally speaking, American children did better in Change (1) - (4), while Japanese pupil did better in Compare (2) - (4). The reason for this could be:

[1] The effect of "manipulative object" is larger in Change problems than in Compare problems.

[2] The wording "less than," which is in Compare (2) problems, is well understood by Japanese children as a typical key word for subtraction.

[3] In the Japanese arithmetic curriculum, Compare (3) is one of the typical examples of addition type called "seeking of large," and Compare (4) is an example of subtraction process called "seeking of small." Children are taught these very well.

2.2. Correct answer percentages between US and Japanese children for second graders in Change (1a), (3), Combine (2a) and Compare (2) differ greatly. American children did better in the former two problems, while Japanese children did better in the latter two problems.

The reason for this could be:

[1] As for (1a), the researcher pointed out in 1.1 that the key word "gave," which implies subtraction, might have worked against Japanese children for their wrong answers. However, Japanese children's correct performance percentage for (1b) was almost at the 100% level, so they are not necessarily weak at problems of Change (1) type.

[2] As for Change (3), manipulative objects seemed to have worked effectively for the performance of American children. On the other hand, Japanese children showed poor results in grasping the problem structure and changing it to computation, as these problems are "addition-converse-subtraction" form, which means that the situation indicates addition while computation is to be induced by subtraction. As explained above, the reason could be proved by the fact that in the wrong answers of Japanese children, there were many children who responded 3 + 5 = 8, thus answered 8; or 3 + 8 = 11, thus 11.

[3] Japanese children's performance results for Combine (2a) are almost at the same level as Change (4a), so it is not surprising. Rather, the reason for the low percentage of correct answers by American children for the same problems could be attributed to some other factor.

[4] As for Compare (2), one of the factors for the low percentage of correct answers might be the key word "less than" which could imply subtraction, as already pointed out in 2.1.[2].

2.3. The case of performance for the third graders showed a similar tendency for both Japanese children and their American counterparts. This suggests that as children advance in age some influences of linguistic and social background decrease.

(3) On relative difficulty

P. Nesher, J.G. Greeno and M.S. Riley worked together cooperatively to study the relative difficulty in addition and subtraction word problems from the viewpoints of empirical knowledge, logical operation and mathematical operations. They indicated the knowledge development stage of children by dividing it into 4 levels. Furthermore, they studied the pupil's ability by applying 14 different types of word problems and examined at which level the children can solve certain problem types. Their conclusion was shown on Table III as their hypothesis (Nesher, et al., 1982).

Table III

Type of Problem	Level 1	Level 2	Level 3	Level 4
Change (1)	x			
Change (2)	x			
Change (3)		x		
Change (4)		x		
Change (5)			x	
Change (6)			x	
Combine (1)	x			
Combine (2)			x	
Compare (1)		x		
Compare (2)		x		
Compare (3)			x	
Compare (4)			x	
Compare (5)				x
Compare (6)				x

Their study has profound logical basis and the level of classification made by them coincides considerably with the results of another research work in the U.S. However, we observe some discrepancy in several points between Table III and data obtained by the author's research in Japan.

Figure 3 shows the sequential order arranged from higher to lower percentages of correct answers to the problems. These percentages are obtained by combining the results of second graders and third graders in Japan. Based on this the author found that it is permissible to classify the relative difficulty for 14 different types of word problems by drawing demarcation lines between Change (2) and Compare (3), Compare (1a) and Change (3), and Change (5) and Compare (5). The results of classification are shown in Table IV. The marks "---> " or " <---" indicate the difference from the classification of Nesher, et al.

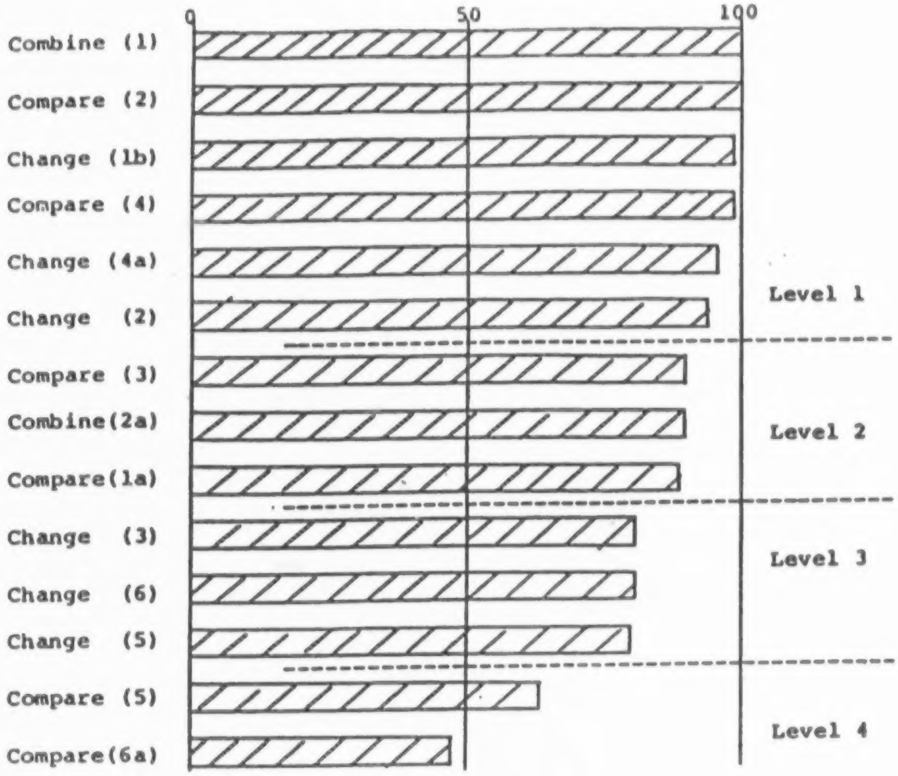

Figure 3

Table IV

Type of Problem	Level 1	Level 2	Level 3	Level 4
Change (1)	x			
Change (2)	x			
Change (3)		------------→	x	
Change (4)	x	←------------		
Change (5)			x	
Change (6)			x	
Combine (1)	x			
Combine (2)		x	←-----------	
Compare (1)		x		
Compare (2)	x	←------------		
Compare (3)		x	←-----------	
Compare (4)	x	←------------	------------	
Compare (5)				x
Compare (6)				x

Now, let us proceed further with this study to see at which level each problem type should be placed, keeping the above difference in mind. Change (3) is classified as level 3. However, when examined in detail, we find they might have been confused by the word "gave" which could imply subtraction, thus they might have applied the "subtraction-inverse-addition" form. Many wrong answers 8 + 3 = 11 could be construed as such. If the sentence of Change (3) problems was altered, such as not including the word "gave," it could be classified as level 2. The author accepts this hypothesis.

Change (4) is placed in level 1 which may possibly be because it is a problem of "subtraction-inverse-subtraction" form. However, the truth may be that among those who answered correctly some might have misjudged the true structure, not having grasped the meaning, but decided their computation as subtraction because of the wording "gave"; thus they got 5 from 8 - 3 = 5 which, incidentally, is the correct answer. If we take those who grasped the true structure of the problem and label them as the sole correct respondents, Change (4) would be classified as level 2.

Combine (2) problems reverse the problem solving process of Combine (1) because, mathematically, it is $a + x = b$ (x is unknown); thus they could be the same level of problem as Change (3) and (4), so it should be classified as level 2, not level 3 as it was classified by Nesher, et al. Compare (2) is classified as level 1, however, from the viewpoint of the empirical development of children and, considering the level difference from Change (2), it would belong to level 2, in the author's opinion.

Compare (3) and (4) would be classified as level 3 in view of the reverse solving process of Compare (1) and (2). However, they should be categorized as changed form of Change (1) and (2) as they could be solved "one more, two more" or "one less, two less." In consideration of these facts, they should be placed in level 2.

To sum up the above results, the reseacher made Table V as a revised version of Table IV.

Table V

Type of Problem	Level 1	Level 2	Level 3	Level 4
Change (1)	x			
Change (2)	x			
Change (3)		x		
Change (4)		x		
Change (5)			x	
Change (6)			x	
Combine (1)	x			
Combine (2)		x	←----------	
Compare (1)		x		
Compare (2)		x		
Compare (3)		x	←----------	
Compare (4)		x	←----------	
Compare (5)				x
Compare (6)				x

Six types of word problems of Change can be mathematically classified as follows:

(1) $a + b = x$ (3) $a + x = c$ (5) $x + b = c$
(2) $a - b = x$ (4) $a - x = c$ (6) $x - b = c$ (x is unknown)

The relative difficulty increases as the number does, and naturally they could be classified as level 1, 2 and 3.

The situation of Combine problems can be categorized as the same level as Change problems and mathematically, (1) $a + b = x$ (2) $a + x = c$ (x is unknown); therefore, they could be placed in level 1 and 2, respectively.

The solving process of Compare (3) and (4) is respectively translated into Change (1) and (2). Therefore, Compare (3) and (4) are one rank higher than Change (1) and (2), and thus could be placed in level 2.

Compare (5) and (6) have the reverse process of problem solving for Compare (3) and (4). Furthermore, the key words "more than" and "less than" are used. Although these two expressions usually imply addition and subtraction, respectively, exactly the opposite computation is required. Therefore, the relative difficulty is the degree of 2 higher ranks and should be placed in level 4.

Examination of such factors as problem situation, problem structure, key words and expressions give Table V some logical basis for the classification of levels. As Nesher pointed out, the pedagogical implications of these analyses are two-fold: "first, one can be more sensitive to the sequence of instruction when one understands the prerequisite knowledge structures for solving certain problems, and can adapt different strategies in teaching at different levels; second, this analysis allows a better understanding of the difficulties that children encounter at different levels of performance" (Nesher, et al., 1982).

The classification of levels by the author needs further refinement for these implications.

4. Conclusion

The analysis made so far and other examination will lead us to the following conclusion:

4.1. As for the relative difficulty in 14 different types of word problems, the empirical data obtained in Japan and other logical study made level classification in Table V, which would show the hypothetical conclusion.

4.2. One of the different points from Nesher, et al., in the classification of levels is of Combine (2), Compare (3) and (4). It is yet to be studied whether these differences come from the educational, cultural or social differences.

4.3. Through the survey, it was made clear that words and expressions, especially key words, could exert influence over the performance of Japanese children.

4.4. The key word will exercise plus effect when they are in conformity with the problem construction. On the opposite case, they will exercise minus effect. Therefore, when teachers instruct children in word problems where these key words are emphasized, their instruction should take into consideration its reverse effects as well as affirmative effects.

Finally, this study was limited by several points. First, the sample size of tested pupils was very small; second, the determination of relative difficulty was rather tentative; third, a fair comparison could not be made because the conditions of testing were not the same between Japan and the U.S. The author would like to continue this study in order to confirm the above conclusions by overcoming these limitations.

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Appendix I   TESTED PROBLEMS

CHANGE

Result unknown

(1a) Takeshi had 5 caramels.
Then Masao gave him 3 caramels.
How many caramels does Takeshi have now ?

$$\boxed{5}\xrightarrow{+3}\boxed{?}$$

(1b) Akiko had 5 marbles.
Then she was given 3 marbles by Satomi
How many marbles does Akiko have now ?

(1c) There were 5 swallows.
4 swallows came flying.
How many swallows are there now ?

(2) Takeshi had 8 caramels.
Then he gave 5 caramels to Masao.
How many caramels does Takeshi have now ?

$$\boxed{8}\xrightarrow{-5}\boxed{?}$$

Change unknown

(3) Takeshi had 3 caramels.
Then Masao gave some caramels to Takeshi.
Now Takeshi has 8 caramels.
How many caramels did Masao give to Takeshi ?

$$\boxed{3}\xrightarrow{+?}\boxed{8}$$

(4a) Takeshi had 8 caramels.
Then he gave some caramels to Masao.
Now Takeshi has 3 caramels.
How many caramels did Takeshi give to Masao ?

(4b) Takeshi had 8 caramels.
He gave some to Masan.
Now he has 3.
How many did he give ?

$$\boxed{8}\xrightarrow{-?}\boxed{3}$$

Start unknown

(5) Takeshi had some caramels.
Then Masao gave 5 caramels to Takeshi.
Now Takeshi has 8 caramels.
How many caramels did Takeshi have in the beginning ?

$$\boxed{?}\xrightarrow{+5}\boxed{8}$$

(6) Akiko had some marbles.
Then she gave 5 marbles to Satomi.
Now Akiko has 3 marbles.
How many marbles did Akiko have in the beginning ?

$$\boxed{?}\xrightarrow{-5}\boxed{3}$$

COMBINE

Combine value unknown

(1) Akiko has 3 marbles.
Satomi has 5 marbles.
How many marbles do they have altogether ?

$$\left.\begin{array}{c}\boxed{3}\\ \boxed{5}\end{array}\right\}\boxed{?}$$

Subset unknown

(2a) Takeshi and Masao have 8 marbles altogether.
Takeshi has 3 marbles.
How many marbles does Masao have ?

$$\left.\begin{array}{c}\boxed{3}\\ \boxed{?}\end{array}\right\}\boxed{8}$$

(2b) There are 6 children on the ground.
4 are boys and the rest are girls.
How many girls are there on the ground ?

COMPARE

Difference unknown

(1a) Akiko has 8 marbles.
Satomi has 5 marbles.
How many marbles does Akiko have more than Satomi ?

$$\begin{array}{c}\boxed{8}\\ | \\ \boxed{5}\end{array}\;+?$$

(1b) Akiko has 8 marbles.
Satomi has 5 marbles.
What is the difference ?

(2) Akiko has 8 marbles.
Satomi has 5 marbles.
How many marbles does Satomi have less than Akiko

$$\begin{array}{c}\boxed{5}\\ | \\ \boxed{8}\end{array}\;-?$$

Compared quality unknown

(3) Akiko has 3 marbles.
Satomi has 5 more marbles than Akiko.
How many marbles does Satomi have ?

$$\begin{array}{c}\boxed{?}\\ | \\ \boxed{3}\end{array}\;+5$$

(4) Akiko has 8 marbles.
Satomi has 5 marbles lessthan Akiko.
How many marbles does Satomi have ?

$$\begin{array}{c}\boxed{?}\\ | \\ \boxed{8}\end{array}\;-5$$

Referent unknown

(5) Akiko has 8 marbles.
Akiko has 5 more marbles than Satomi.
How many marbles does Satomi have ?

$$\begin{array}{c}\boxed{8}\\ | \\ \boxed{?}\end{array}\;+5$$

(6a) Takeshi has 3 marbles.
Takeshi has 5 marbles less than Masao.
How many marbles does Masao have ?

$$\begin{array}{c}\boxed{3}\\ | \\ \boxed{?}\end{array}\;-5$$

(6b) Takeshi has 3 caramels and
5 less than Masao.
How many does Masao have ?

Appendix II

TABLE

Aspects of development (from P.Nesher ,1982)

The level	Empirical knowledge	Logical operations	Mathematical operations
1 Counting sets	Reference to sets, adding and removing members to sets. Understanding 'put' 'give', 'take', etc. as denoting change in location or possession	$x \in P$ $x \in R$ $n(P)$ $n(R)$ $H(a, b)$ ('Have')	Ability to count and find the cardinal number of a set. The order among numbers. $2 < 5 < 8$
2 Change	Ability to link events by cause and effect. Reference to the amount of change. Understand sequence of events ordered in time in a non-reversible manner	$G(J, o)$ becomes $C(H^{-}(J, o))$ $R(J, o)$ becomes $C(H^{+}(J, o))$ when: 'G': 'give' 'R': 'receive' 'C': 'cause'	Understanding addition and subtraction as procedures. '+' and '–' are distinct $a + b \rightarrow c$ $a - b \rightarrow c$
3 Part-Part-Whole	A reversible part-part-whole schema is available, and can be used to find unknown part in any slot in a sequence of events. Understanding class-inclusion.	Understanding the additive relation among three sets (P, Q, R). $P \cup R = Q$ $P \subset Q$ $R \subset Q$ $P \cap R = 0$ $x \in (P \cap Q)$ $n(P) + n(R) = n(P \cup R) = n(Q)$ $n(Q) - n(P) = n(R)$	Understanding the relation among three numbers in an equation (=). Connection between addition and subtraction: if $a + b = c$, then, $c - b = a$ and $c - a = b$
4 Directional relations	Reversibility of non-symmetrical relations. Ability to handle directional descriptions (more/less), and quantify a relational set (relative comparison).	$R(a, b) = R^{-1}(b, a)$ if $m < n$, then, $n > m$ Coordination of $n(a)$, $n(b)$, $R(a, b)$ and $n(a - b)$.	The ability to handle inequality and its relationship to equality, equalizing it by addition or subtraction: if $a > b$, then $a - c = b$ and $b + c = a$

Discussion of Professor Ishida's Paper

Silver: Thank you very much for a very interesting presentation. It is now time for questions. Mr. Ishida would like to receive the questions in Japanese, so if the Japanese would like to ask questions, do so in Japanese and they will be translated into English for the rest of us. When the Americans ask questions, please ask them so they can be translated into Japanese. Let me start off.

In the United States, it is not unusual for teachers to teach children to look for key words like "left," "left over," "remaining," and so on as words which would cue the appropriate operation to be used to solve the problem. Is this typical of Japanese teachers as well?

Ishida: In my view, some teachers teach children to look for key words and others do not, so there does not seem to be a typical pattern in Japan. But, in the lower grades, many children notice key words themselves, after they have experience solving many word problems. Then, they sometimes make mistakes in choosing an operation based on key words.

Clarkson: Yesterday, when we were talking about the textbooks, I gave information that a large number of word problems in U.S. textbooks have key words in them, and that it seems to become more prevalent in the upper grades. Do most word problems in the Japanese textbooks have key words?

Ishida: On matters of adding or subtracting in the lower grades they do have a lot of key words such as "in all," "total," "add," "are given," "remain," "difference," "spend," "take away," etc. As subject matter gets more difficult they don't necessarily treat problems in terms of key words.

Shimada: Some time ago I heard a comment on sentence structure used in arithmetic problems for lower grades by a linguist. His comment had to do with tense in different sentences in one problem. It would be easy for children to follow if the tense of successive sentences is in the natural order, for example, the first one past, the second one half past, and the third one present. But if successive sentences are in an order different from the natural one, this would be a source of confusion for the children. And such a misleading order of tense is likely to happen in describing problem situations for inverse operation. This is not an opinion of a mathematics educator, but of a linguist. Do you feel it is convincing?

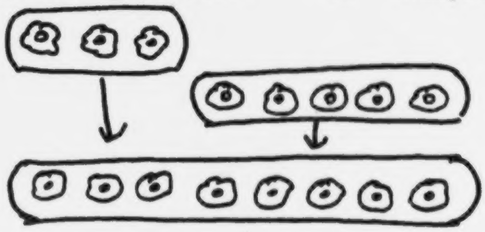
Becker: To the extent the problem exists, if the problems were represented in a concrete manner with the children (that is, if they were acted out or if physical objects were used) I wonder whether that might help to minimize that kind of linguistic difficulty. I have read accounts of this in some of the literature. But I have no idea of the extent to which this happens in Japanese primary schools. But I am quite sure it doesn't happen very often in American schools.

Clarkson: I didn't understand, so I need a translation in English. What doesn't happen in U.S. schools? The confusion over the tenses?

Becker: No, what I meant to say is that, in general, teachers do not make concrete representations of the problem. As a little bit of informal data about this, at my university we have, perhaps, the largest elementary teacher education program in the state of Illinois (which is one of the largest states in the United States). In our elementary math methods course we place a lot of emphasis on the use of concrete manipulative aids. When these students go out to do their student teaching, however, they find that their supervising teachers are unfamiliar with these things.

Ishida: Although this came up in Professor Sugiyama's presentation yesterday, the use of manipulative objects is becoming more and more common in Japanese elementary schools. For example, consider the following:

Akiko has 3 marbles.
Satomi has 5 marbles.
How many marbles do
they have together?



Manipulative methods like this are considered to be effective teaching for conceptual understanding and problem solving.

Clarkson: Related to that, most of the pictures in the newer textbooks in the U.S. are related directly to the problems and many also present manipulative materials.

Travis: I noticed in your paper that some problems were more difficult for the American children when manipulatives were used. Do you have some hypotheses about which problems are best enhanced through the use of manipulatives and which ones manipulatives may actually impede the instruction?

Ishida: Please look at Table I and Table II. Table I indicates the result of the manipulative objects being provided to the American children, while Table II is that of not using manipulative objects.

The difference of the correct percentage of Change problems are greater than those of Compare problems. So, I think Change problems are enhanced better through the use of manipulative objects.

Wilson: You made an assessment of these children's performance and he provided them, so we really shouldn't claim that the instruction has been done with manipulatives. We don't really know what background these children had before he sat them down with objects and gave them these problems.

Silver: As a comment, there are also data from the studies by Carpenter, Moser and Hiebert with kindergarten and first grade children who had not received specific instruction in the methods of solving these problems. They made very frequent spontaneous use of concrete materials to model the problems. Their modeling conformed very nicely, in general, to the structure of the problems, and the actions the children performed on the objects conformed to the actions represented in the problems.

Becker: I just want to follow up on what Ed said. Vaguely I recall that, either in your paper or perhaps I read it somewhere else, that in the Carpenter, Moser, Hiebert research, at a different time when they gave instruction on how to solve those problems, the kids, in fact, then had more difficulty. Is that correct?

Silver: Yes, in follow-up studies with some of the children, after they had been taught to solve the problems using equations and to write equations as the model rather than acting the problem out, the performance level decreased. But the flaw in that analysis is the lack of a long term follow-up to see whether or not that was a temporary decrease in performance. We don't know whether the new skill of writing equations, which was being learned, might have caused a decrease in performance.

Shimada: The point dealing with teaching materials has come up the past couple of days. I would like to clarify the situation in Japan, as to their use. There are two kinds of materials for teaching arithmetic: manipulative materials such as marbles, magnetic buttons, beads, rods, and so on; and structured materials such as Ciusenaire rods, Diene's blocks, and Stern's materials, all of which have some built-in structures. In the case of structured materials, they are used eagerly by some groups of teachers, but not at all by others. Manipulative materials are used more widely. They are used especially in teaching the meaning of fundamental operations, such as addition or subtraction, in order to demonstrate in a concrete fashion that seemingly different problem situations are reduced to the same relation between numbers. Though there are not a lot of teachers who use the key word approach, there is a general tendency to emphasize the use of manipulative materials.

Travers: Yesterday in a comparative study of American and Japanese books, we heard that in the early grades there is an emphasis on meaning and understanding before computation in Japan; whereas, in the United States the focus seems to be on procedures (that is, algorithms without understanding). These data here, though based on small samples, seem to be in conflict with that observation from yesterday.

Ishida: Japanese teachers surely emphasize the meaning of operations before computation. But I do not understand why my data seem in conflict with that observation.

Clarkson: Yesterday, Professor Sugiyama's remarks were based on some first and second grade tests that he had seen that had no word problems in them. So he was reporting that in the U.S., we were dealing strictly with the manipulation of numbers separated out from verbal situations. What he reported for that particular textbook was true. Again, in the newest textbooks there is more of an attempt to put simple verbal situations into the lower grades.

Silver: If I may add a comment. It seems that when we attempt to teach the meaning of operations, we use situations as a vehicle of communication. Those situations need to be communicated themselves in language. In problem situations, the nuances of language and confusions that occur within the language then reverberate confusion relative to the mathematics.

Kantowski: I have a comment related to that. On page 260 , Professor Ishida states that until the children's ability advances to the stage where they start to realize the problem structure, a simpler form of expression could help them to solve the more difficult type of word problem. It seems to me that he is saying what Dr. Silver has just said. In dealing with the understanding and meaning of the mathematics first, and ensuring that the children understand the subtraction or the comparison or the change before changing the wording, we would be more able to help the children deal with the changes in wording of the problem later. In fact, instead of trying to do the two things at once, the first thing to deal with is the mathematics expressed in easier language; then, in fact, the language could be changed if the children understood the mathematical structure.

Rachlin: Paul Cobb reported on a study at AERA this year. I'd like to refer to one of your items and then talk about what his study was like. It is under the change items in Appendix 1 and item #2 under "change" on page 271, the fourth item down: Takeshi has 8 caramels, then he gave 5 caramels to Masao, how many caramels did Takeshi have now? What happened in Cobb's study was that he had worked with 34 first grade students, and in working with the students on a task like this, he found that what the children did was to take the 8 caramels that were Takeshi's and then go back to the bag where the caramels were and take 5 more caramels and give them away. The error that came is that for us we knew that the child was supposed to give Takeshi's candy away, but eight out of the 34 children didn't do that. Instead they are used to a situation where they may only have five toys that they are playing with now, or eight caramels that they have right now, but there are more on the side and if I must give you some, I'm not going to give you the ones I'm using, I'll give you some from my bag and then you have some too. In the case of Cobb's study he was suggesting that if there was instruction the children might understand better what the task was and once they understood it there would be different results that would occur. Have you had any kind of similar reactions in the instruments?

Ishida: I have seen also such a situation on teaching first graders. I feel that such results come from the conventions of making the word problems. When we write "Takeshi had 8 caramels. Then he gave 5 caramels to Masao," we mean that Takeshi gave 5 caramels out of 8 caramels which were referred to in the first sentence. But children do not always comprehend such meaning. We use abbreviated sentences by convention and confusion may arise from them. What I am saying is that sometimes if you make the word problem too clear mathematically in terms of language, the children may lose some of the realistic situation and then answer in terms of the realistic situation.

Wilson: There are really two questions. I think that Professor Travers' question was not answered because we got off into whether or not we interpreted some of yesterday's discussion correctly. The question deals with your classification based on your view of meaning of operation for children. You come up with a different classification on the basis of your data, but is that interpretation of the meaning of operation a part of learning to answer questions and problems like these? My second observation or second question would be to point to Nesher, Greeno, and Riley from whom you draw this data and assert that they probably do not think about meaning of operation at all. To them understanding mathematics, understanding arithmetic, understanding the operations of addition or subtraction is a procedure, and the meaning of operations in some problem or mathematical sense is not part of their way of thinking. Thus, the first question is: Does he interpret his reclassification of items on the idea of meaning of operation? A mathematical basis for the operation? The first question is about the meaning of operation.

Shimada: But actually carried out data touches on the essence of meaning.

Wilson: Okay, that's the answer to my question. No, it is not. But we did say that when the question was raised before.

Ishida: I am sorry, I cannot understand your question correctly. I categorized the meaning of operation by situations and structures in word problems, depending on Carpenter, et al. Their classifications are clear. So relative difficulty levels on the basis of my data are reasonable and meaningful, though my sample is small.

Dealing with small numbers like 8, 5 and 3, because I have concern with whether the children got the meaning of the operation and also I do not want them to get into trouble with actual calculation.

Silver: We have to break for coffee. We can continue conversation over coffee.

Professor Rachlin's Paper

Sawada: I want to introduce to you Dr. Rachlin. Dr. Rachlin is Associate Professor of Mathematics Education at the University of Hawaii. He was at the University of Georgia as a Teaching Professor in 1981 and also has many publications. For example, the NSF algebra work from 1985 to 1988 and also Director of the project on teaching problem solving in the algebra curriculum from 1984-85.

ISSUES OF LANGUAGE ON WORD PROBLEMS IN THE UNITED STATES OF AMERICA

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> Once something is acclaimed it can be ignored in a noble way. The acclaim which we suffered was that each reader-teacher picked the part he liked best and proclaimed it was exactly what he was doing!
>
> -Jerome S. Bruner,
> The Process of Education Revisited
> Phi Delta Kappan, September 1971

Introduction

In the United States, the last decade saw "problem-solving" become, if not a household word, then a classroom word. Mathematics was but one of the subject areas calling for an increased emphasis on problem-solving behaviors and skills. Teachers from pre-school to university began to explore and attempt to understand this new watch word. For some teachers, problem solving was the four-step solution to motion, mixture, and work problems. For others, it was the "Think" exercises found in the old elementary school texts.

The National Assessment of Educational Progress (NAEP, 1979) confirmed that students' problem-solving performance was weak. The NAEP Panel to Interpret the 1977-78 Assessment of Mathematics Data recommended that "teaching approaches strive to promote understanding of concepts and to link skills to problem-solving activities." But despite the call for increased emphasis on problem-solving performance expressed by the NAEP Panel, as well as by the National Council of Teachers of Mathematics and the National Council of Supervisors of Mathematics, there has been little change in students' problem-solving performance. The only problem-solving area that showed significant growth in the 1982 National Assessment (NAEP,

1983) was in 13-year olds' solutions to routine word problems (those usually found in textbooks and practiced in school). And even this success may be more a product of students having learned to take the two given numbers and pick an operation to use with them. What is credited as problem solving may in fact be new skills.

Just as Bruner lamented that discovery learning was "acclaimed into impotence," so the trend towards problem solving in the United States suffers from the ability of teachers, curriculum developers, and researchers to fit the magic words "problem solving" into their syllabi, texts, and proposals without changing their philosophy toward the learning of mathematics. In many schools, problem solving is being taught rotely. Textbooks and journals refer to the same "non-routine" tasks. Work, mixture, and motion problems have been joined by "handshakes," "pigs and chickens," and "squares on a checkerboard" problems. Although these "non-routine" tasks are entering the curriculum, they do not necessarily promote understanding of concepts or link skills to problem-solving activities. This is not to say that there are not some excellent programs throughout the United States that do in fact organize the mathematics curriculum around problem solving. There are! They are generally found at the elementary school level and build on a Nuffield, Wirtz, Workjobs style inquiry approach. Their success relies heavily on the educational philosophy and mathematical awareness of the teacher. It is rare to find a program at the high school level which truly organizes its curriculum around problem solving.

Common among the inquiry approaches of the elementary school is a strong emphasis on understanding of problem situations before problem solving is begun. The fascinating dilemma is that students who can solve simple word problems when they are stated orally, or even create and solve their own story situations, will have limited success when similar problems are presented in a written format. It is as if the written tasks are no longer meaningful. But perhaps, for the child, there is no reason to assume that the two tasks should be connected. In the following account, my son Jeff was just beginning second grade.

> Beside Jeff´s bed hung a chalkboard. If I had been a physical education teacher, I probably would have placed a trampoline there, but since I am a math teacher, a chalkboard won out. Every so often I would come into his room and write a math problem on his board. At another time, he would come in and solve the problem. Still later, I would stop back and check his solution. If it was correct, I would erase it and place another problem on the board. If it was incorrect I would call him in and we would discuss it.
>
> On one occasion, when Jeff had just been introduced to addition with regrouping (carrying), I wrote the problem:
>
> $$\begin{array}{r} 24 \\ +16 \\ \hline \end{array}$$
>
> When I later returned to the room, I found Jeff´s solution:
>
> $$\begin{array}{r} 1 \\ 24 \\ +16 \\ \hline 41 \end{array}$$
>
> After I called Jeff into the room, the following dialogue ensued.
>
> "Jeff, I think there´s an error here."
>
> "No, there isn´t Dad. Look! Four and six are ten. You put down the one and carry the one. One and two are three and one is four. The answer is 41."
>
> "No, Jeff, I think something is wrong here."
>
> "Look! (Jeff spoke a little louder to make his explanation more acceptable.) Four and six are ten. You put down the one and carry the one. One and two are three and one is four. The answer is 41. You ask Miss Frame, she´ll tell you how to do these."
>
> As a math teacher I took this as a sign that a concrete embodiment was needed. After all, using concrete objects makes math make sense. I got a pack of toothpicks and a box of rubber bands and sat on the floor beneath the chalkboard. Jeff had grouped by tens before and had no difficulty representing 24 as two tens and four ones and 16 as one ten and six ones. He added (combined the two piles) and got three tens and ten ones or after trading in the ten ones for one ten he had an answer of 40 with the toothpicks. At this point he looked back and forth at his pile of toothpicks and the chalkboard. Finally, he said very seriously, "That´s what you get when you add toothpicks, but when you work on the board you get this answer."

I later learned that Jeff´s response is not that unusual. It has been reported by other parents and teachers (Wirtz & Kahn, 1982). Why should children expect that what they get on the board should match what they get with objects? If we are using concrete objects to serve as a foundation for arithmetic operations, we must make sure that the procedures used with the concrete materials parallel the procedures used in the rote algorithms being taught. A parallel statement may also be true for word problems. Why should students expect that the *meaningful* problems that they create bear any resemblance to the *arbitrary* and contrived word problems in the text?

> Help Mary find her age: If Mary´s 31-year old mother is three more than twice Mary´s age in four years, what is Mary´s age?

As one newspaper columnist put it, "If Mary doesn´t know her own age, why should I tell her!" It is little wonder that by college, students meaninglessly attach variables to the infamous "students and professors" problem. No attempt was made by over 25% of the calculus-level students to make sure they understood the task either before *or after* they wrote 6s = p to represent a situation in which there are six times as many students as professors (Clement, Lochhead, & Monk, 1981). On the other hand, Mayer (1982) reports that when students were asked to translate relational statements like this one into a computer program, the error rate fell dramatically. Students must be lead to appreciate the value of understanding a problem before attempting to solve it.

What is it about word problems that makes them so difficult to understand? Threadgill-Sowder and Sowder (1982) suggest that it may be *the words*. They found that drawn versions of traditional mathematical story problems resulted in superior problem-solving performance for 262 fifth graders. Kulm and others (1974) found the opposite to be true. In a study with 116 first-year algebra students, a pictorial version of algebra word problems was found to be more difficult than either a textbook version or a student version. The hidden catch is that to teach problem solving we cannot take the problem out of the situation. Once a student can immediately apply a procedure to complete a task, it no longer poses a problem. The role of instruction in problem solving is to help students develop their own approach to the phases of problem solving.

The Language of Word Problems

Perhaps the most thorough analysis of the effects of the language of word problems is given by the monograph, Task Variables in Mathematical Problem Solving, produced by the Task Variables Working Group of the Georgia Center for the Study of Learning and Teaching Mathematics (Goldin & McClintock, 1979). In the opening chapter of this monograph, Kulm (1979) identifies four major categories of variables related to the statement or presentation of a problem:

- variables which describe the problem syntax,
- variables which characterize the problem's mathematical content and non-mathematical context,
- variables which describe the structure of the problem, and
- variables which characterize the heuristic processes evoked by the problem.

Syntax

These variables provide ways in which a teacher or researcher might analyze the complexity of a proposed problem. Syntax, the arrangement and the relationship of words, phrases, and symbols in problem statements, is known to affect problem solving (Barnett, 1979). Syntax variables such as problem length, grammatical complexity, and data sequence are explicit and readily susceptible to counting. Kulm (1979) notes that although little research has been done on the effects of variations in syntax on problem-solving behavior processes, in two tasks with identical context, a slight change in the syntax may produce a great change in the way students solve the problem. As examples, Kulm presented the following two problems from Krutetskii (1976):

> A horse moved at a speed of 12 km per hour for half the time spent on a journey, and at 4 km per hour for the rest of the time. Find the horse's average speed.
>
> A horse traveled half a journey at a speed of 12 km per hour, and at 4 km per hour for the rest of the journey. Find the horse's average speed.

What may at first seem like unimportant changes produce drastic changes in the problem and the operations used to solve it.

A change as minor as the position of the unknown quantity in a missing term task can greatly affect a student's solution process. Wagner, Rachlin, and Jensen (1984) interviewed ninth grade algebra students in Athens, Georgia and Calgary, Alberta with a series of problems based on variations in the missing terms of the form $\square * b = c$ or $a * \square = c$ where * represents an algebraic operation and a, b, and c are whole numbers, fractions, polynomials, algebraic fractions, or radical expressions. They found wide differences in the ease with which students were able to solve these missing term tasks depending on the operation substituted for *. The relative difficulty of the problems varied depending on the placement of the missing term. For example, the following missing term task was a problem for most ninth grade algebra students:

What number multiplied by 2/3 equals 3/2?

Many students simply multiplied 2/3 x 3/2, while others were unsure whether to represent the problem as 3/2 ÷ 2/3 or 2/3 ÷ 3/2. The students were quite rule-oriented and liked to state generalizations about the solution process before beginning the problem. In many cases, students falsely generalized about how to do the problem, either over-generalizing about something that occurred in the problem or something they heard their teacher say. For example, when given the problem:

1/4 subtracted from what number gives 7/12?

one girl responded, "Always do the 'opposite' of what it says in the problem." In this case, her procedure worked and served to reinforce her misconception that the opposite operation will always yield the missing term (Jensen, Rachlin, & Wagner, 1982).

Hiebert (1982) examined the effect of the position of the unknown term on first-grade children's representation and solution processes for verbally presented addition and subtraction problems. Forty-seven first-grade children were given three joining problems and three separating problems in an individual interview. As with the algebra students, Hiebert found that the position of the unknown had a profound effect on the solution processes and relative difficulty of the task. Fifty-five percent of the responses to the verbal problems $a + b = \square$ and $a - b = \square$ included modeling with cubes. The percentage drops to about 40% for the $a \pm \square = c$ problems and to about 18% for the $\square \pm b = c$ problems. The latter problems were rarely modeled with cubes. Some researchers (see for example Cobb, 1986 and DeCorte, Verschaffel, & DeWin, 1985) have questioned the meaning of tasks such as the following example of an $8 - 3 = \square$ word problem used by Hiebert.

> Bill had 8 marbles. He gave three marbles to Susan.
> How many marbles did Bill have left?

These researchers argue that the students do not truly understand the intent of such problems. For example, Cobb reported that on a similar task eight of 34 first graders would act out the problem by first counting out 8 marbles and then reach into the bag for more marbles as they heard the second sentence. Interviews revealed that these children assumed that the entire bag of marbles belonged to Bill. Although several children might play with marbles and be the temporary owner, the toys usually belong to one or more of the children who are sharing them with the others. The interpretation that Bill gives himself 8 marbles and Susan 3 marbles to play with yet Bill still has 8 marbles with which to play, is consistent with their real-life experiences. In a sense, these typical story problems are somewhat artificial. Although this analysis may suggest one source of error, a greater one occurs in the realm of content domain. As the content domain becomes less familiar, errors in the applications of algorithms become more likely.

Mathematical Content and Non-Mathematical Context

Webb (1979) provides a detailed categorization scheme for the content and context variables of a problem. The content of a problem is the mathematical meaning of the task. Often content refers to a particular content area. In his paper, Webb develops four main subdivisions for content variables:

- the mathematical topic,
- the field of application,
- the semantic content, and
- the problem elements.

By semantic content, Webb is referring to the meanings of mathematical key words or technical words or phrases in the problem statement. Problem elements refers to the phrases in a problem statement which contain essential items of information such as givens, allowed operations, and goals.

In the study described earlier, Jensen, Rachlin, and Wagner (1982) designed a series of tasks to examine the influence of the content domain on the students' processes of solution. Although these tasks were syntactically parallel, the ways in which the students solved the problems changed as the content areas became more complex.

1. What number multiplied by 17 is 204?
2. What number multiplied by 2/3 = 3/2?
3. What binomial multiplied by $x + 5$ equals $2x^2 + 15x + 25$?
4. What polynomial multiplied by $x + y$ equals $2x^2y + 2xy^2 + 3xy + 3y^2$?

Almost all the ninth grade algebra students solved Task 1 dividing 204 by 17. Yet many students attempted to solve Task 2 by repeated "guess and test." The students' inability to solve this task was frequently due to a fixation with one approach toward a solution. For example, one student of above-average ability wrote 2/3 x n = 3/2. Then he reasoned that this was the same as 4/6 x n = 9/6. Since 6 x 1 = 6, he reasoned that the denominator of the missing fraction is 1 and the numerator is the mixed number 2 1/4; that is,

$\frac{4}{6} \times \frac{2\ 1/4}{1}$. Next, he rewrote $\frac{2\ 1/4}{1}$ as the complex fraction $\frac{9/4}{1}$

Since he had never experienced complex numbers before, he tried the problem again. This time he wrote 20/30 x n = 45/30. Once he realized that his answer would again be the same complex fraction, he gave up on the problem. Even though he read the complex fraction as 9/4 divided by 1, he made no attempt to solve the problem by division or to set up a new subgoal of determining the generalization for division by 1.

Rachlin (1982a) included a series of parallel missing-term tasks from different content domains in his study of the mathematical understanding of four successful college-level intermediate algebra students. As in the study with ninth grade students, he found wide variability on syntactically similar tasks. For example, the college students experienced a great deal of difficulty with the following real number task.

What real number added to the $\sqrt{3}$ is equal to $\sqrt{6}$?

Although the students had received either an A or a B on their chapter tests on operations with real numbers, they sensed a lack-of-closure in writing their answer as $\sqrt{6} - \sqrt{3}$. Students would try anything to "finish" the problem. Some tried fractional exponents, others changed the index, and still others turned to square root tables in an effort to get "the answer." Students have learned to accept fractions as both indicated operations and numbers, but real number expressions still feel incomplete.

In addition to the subdivisions for the content variables, Webb (1979) delineated three main subdivisions for the form or context of a problem. These include:

- the problem embodiment or representation,
- the verbal context or setting, and
- the information format.

It is not clear that using real world problems will help improve students' problem solving. Travers (1965) found that although ninth grade boys preferred social-economic, mechanical-scientific, and abstract problem-solving situations in that order, the problem-solving success of students on these tasks was no greater. And what is real world for one student may not be for another. Brownell and Stretch (reported in Webb, 1979) found a significant increase in difficulty as the familiarity of the problems decreased.

Structure

While syntax variables tend to be quantitative measures used to describe a problem, and content and context variables tend to be descriptive measures of the surface features of a problem, structure variables describe the underlying mathematical characteristics of a task. How students perceive a task shapes the other processes which they may bring to bear on the resolution of a task. The various solution paths which a student selects establishes the structure for the problem. For example, the task "What number divided by 24 equals 3/4?" has a wide variety of appropriate structures depending on the way in which the task is perceived; e.g., as equivalent fractions, a proportion, a division problem, an equation, etc.

What is the relationship between a given student's structure for problems and his or her teacher's anticipation of the student's structure? Wagner, Rachlin, and Jensen supplemented their study of students' learning difficulties in elementary algebra with the students' classroom teacher's analysis of the students' problem-solving processes in algebra. After eight interviews were conducted with each of 10 students in Athens, Georgia and 4 students in Calgary, Alberta, the classroom teachers were asked to complete all of the interview tasks. Then they were requested to guess how each of their students would solve the problems. Finally, the teachers were able to listen to (in Georgia) or watch (in Alberta) the interviews to test the accuracy of their predictions. Rachlin (1982b) reported on results of the interviews with the Calgary teacher. The teacher was very flexible with tasks such as "What number added to the sum of 17 and 6 equals 6?" and could solve the tasks in several ways. But, he was surprised to find that only one student solved this problem by recognizing that because of the sixes, all that the student needed to find was the number that added to 17 equaled 0. The other students first added 17 and 6 and then subtracted 23 from 6.

The structure of the students' solutions also varied at times from classroom practice. For example, the teacher was surprised that three of the four students solved tasks such as "What trinomial subtracted from $5x^2y - xy^2 + 7$ equals $-x^2y + 8$?" by writing the parts vertically. With regards to the vertical form the teacher commented, "They've seen it occasionally in the textbooks, but I've never assigned the problems that have vertical form." In fact, for the three years that the teacher had these students in class, he avoided writing any polynomial addition or subtraction tasks vertically because he felt that writing them both horizontally and vertically would be confusing.

Heuristic Behavior

Kulm (1979) listed heuristic behavior as his last set of variables for the language of problems. Some problems, it is argued, cry out for the use of heuristics. Still it is hard to separate the learner from any such analysis. For example, Eisenberg and Dreyfus (1985) reported on a study in which 32 college level students in the United States and Israel were asked to solve the following two problems.

> Tennis Player Problem: 463 tennis players are enrolled in a single elimination tournament. They are paired at random for each round. If in any round the number of players is odd, one player receives a bye. How many matches must be played in all rounds of the tournament together to determine the winner?
>
> Given n points in the plane, not all collinear, show that there are (at least) n straight lines each containing two or more of the n points.

In reporting the results of their study Eisenberg and Dreyfus begin by stating that all background information on the students can be ignored. Regardless of their backgrounds, the students were consistent in the following behaviors:

(1) They rushed toward an answer, often bypassing a rational analysis of the givens.

(2) They used known procedures uncritically.

(3) They seldom thought whether alternative routes were available before attacking a problem.

(4) They did not recapitulate nor check whether their answers were reasonable.

(5) They did not generalize unless they were explicitly required to do so.

As a consequence of (1), (2), and (3), Eisenberg and Dreyfus report that elegant solutions were almost entirely missing. In fact, no similarities in either the process or the intent of the problems were perceived; each problem was approached as a unique task. Although it may be possible to find problems that are ripe for the use of heuristics, we will also need to prepare students to be problem solvers. Curriculum materials need to be developed which help students develop problem-solving behaviors, as they learn the high school mathematics curriculum. Occasional side trips into the world of non-routine problems and real world problems must be coordinated with a heavy introduction to problem-solving processes in the child's own world--the world of the classroom.

The Hawaii Algebra Learning Project

One example of such an endeavor is the Hawaii Algebra Learning Project currently in progress at the Curriculum Research and Development Group of the University of Hawaii. The goal of the Hawaii Algebra Learning Project is to develop process-oriented algebra curriculum materials appropriate for the majority of high school students. The theory being tested is that the successful study of algebra requires a set of problem-solving processes that are usually not sufficiently developed in prior mathematics courses. Three of the basic processes identified as starting points are included in Krutetskii's (1976) model of mathematical abilities--generalization, reversibility, and flexibility.

Central to the experimental methodology used in the Hawaii Algebra Learning Project is the bringing together of researchers, teachers, and evaluators to study the flow of algebra instruction in the classroom. Clinical interviews examine the development of the thought processes of above-average, average, and below-average students as they "think aloud" while solving interview problems, sample homework problems, and problems from the day's lesson. The classroom lesson plans are modified daily, based on the research team's analysis of the interviews. The evolving method of instruction attempts to overcome some of the difficulties students have in solving standard and nonstandard algebra problems that range across a content x process x form matrix - (integers, fractions, real numbers, polynomials, algebraic fractions) x (flexibility, generalization, reversibility) x (expression, equation, graph).

Standard problems are used as foundation tasks upon which generalization, reversibility and flexibility tasks are constructed. For example, the following sample tasks involve polynomials operations and equations:

	Algebraic Expressions	Equations
Standard	Multiply: $(2a+3)(2a-3)$.	Solve: $4a^2-9=0$.
Reversibility	Find the binomial which multiplied by $2a - 3$ equals $4a^2-9$.	Find an equation whose solutions are $\pm 3/2$.
Generalization	Find 2 binomials whose product is a binomial... a trinomial...has 4 terms...has 5 terms.	Find a quadratic equation whose solutions are proper fractions.
Flexibility	Find the binomial which multiplied by $2a-3$ equals $4a^2-9$. Can you find the binomial another way?	Solve: $4a^2-9=0$. Solve: $4(a+1)^2-9=0$. Solve: $4(2a+1)^2-9=0$.

Psychological Processes in Learning Algebra

A basic premise of the Hawaii Algebra Learning Project is that the learning of algebra, beyond the level of rote memorization of formulas and algorithms, can be regarded as a kind of problem-solving process. That is, even the application of formulas to "routine" textbook exercises involves some degree of problem-solving activity on the part of most students, at least for a while. Thus, the project alters classroom curriculum and instruction in elementary algebra by using the problem-solving processes of generalization, reversibility, and flexibility, to guide the initial selection of tasks.

Generalization

Krutetskii (1976) considered the ability to generalize mathematical material to be on two levels:

> (1) a person's ability to see something general and known to him in what is particular and concrete (subsuming a particular case under a known general concept), and
>
> (2) the ability to see something general and still unknown to him in what is isolated and particular (to deduce the general from particular cases, to form a concept). (p. 237)

The first of these levels has been characterized by Dienes (1965) as an extension of an already-formed class. This notion of generalization is commonly reflected in the ordered series of exercises found in most mathematics texts in which increasingly more complicated extensions of a form are made. Graded sequences of problems within a topic and similar forms of problems across topics are included in the course materials to develop this aspect of generalizing. For example, the following series of problems shows the generalization of the concept of addition across integers, fractions, polynomials, algebraic fractions, and real numbers.

(1) Find three integers whose sum is -2.

(2) Find two polynomials whose sum is $5x^2+2x+4$.

(3) Find two fractions whose sum is 3/8.

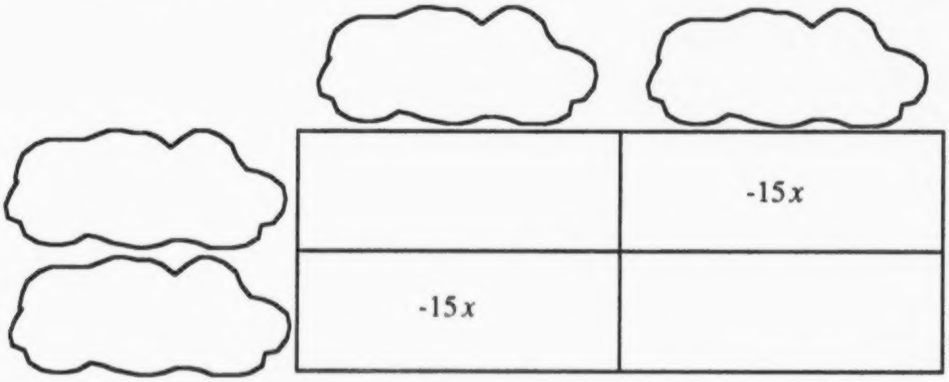
(4) Find two fractions whose sum is $\frac{2x + 7}{8}$.

(5) Find two real numbers whose sum is 12.

The variety of solutions given in class are used to challenge the students to give more complicated solutions. As they search for solutions, the above-average students stretch to increase the range of acceptable solutions and the average and below-average students follow this lead by creating new examples of the enlarged generalizations. Tasks without unique solutions are used in class to develop the adolescents' ability to generalize their solutions to unknown variants. For example, students are guided on problems like problem 5 to find pairs of numbers from the naturals, integers, rationals, and irrationals. In addition to particular solutions, students are encouraged to offer general solutions.

Krutetskii's second level of the ability to generalize mathematical material is closely related to Dienes' (1965) definition of abstraction as a process of class formation. For example, the concept of a trinomial square was developed through discussion of open-ended tasks such as:

> Fill in the table below so that you end up with a trinomial square.



> Is your answer unique?

Reversibility

Krutetskii (1969) considers reversibility an essential aptitude for the formation of algebraic concepts. He defines the basic concept of reversibility as follows:

> By reversible (two-way) associations (and series of associations) we mean those associations in which the thought or realization of the second element (or of the last element) evokes the thought or realization of the first element. (p. 51)

In a more general sense, he explains reversibility as "an ability to restructure the direction of a mental process from a direct to a reverse train of thought" (Krutetskii, 1976, p. 143). For example, in the expression $a + b = c$, we might be given values for a and b, and be asked to find a value for c. The reversibility of this addition incorporates three variations: where the values of a and c are given and the value of b is to be found, where the values of b and c are given and the value of a is to be found, and where the value of c is known and both a and b are to be found. For children to possess complete reversibility of addition of whole numbers, they should be able to solve problems involving all three variations:

$$5 + \triangle = 7,\ \square + 2 = 7, \text{ and } \square + \triangle = 7.$$

Correspondingly, a student who possesses complete reversibility of addition of polynomials should be able to solve the following three tasks:

(1) What polynomial added to $5x^2+3xy$ yields $3x^2+y^2$?

(2) The trinomial $2x^2-3xy+y^2$ added to what polynomial yields $3x^2+y^2$?

(3) Find two polynomials with at least one nonsimilar term such that their sum is $3x^2+y^2$.

Flexibility

Flexibility was identified by Krutetskii (1976) as the ability to switch from one level of thinking of a problem to another. Kilpatrick (1978) noted that there were two aspects of flexibility included in Krutetskii's research: flexibility that can be shown either within or across problems. Within problem flexibility refers to the ease with which a student switches from one method of solving a problem to another method of solving the same problem. In the Hawaii Algebra Learning Project, problems such as the following are used to encourage the students to develop alternative ways to solve the same problem. Students learn to solve linear equations through guess and test, exhaustive testings with a computer program, drawing a diagram, working backward, and via the traditional algebraic approach of creating equivalent equations.

a. Solve the following equation for x: $7 - 5x = 32$.

b. Solve the equation above another way.

c. Write an equation like the equation above that has a solution of -4.

The ability to switch from one approach to another, more efficient, approach is a question of degree. Across problem flexibility refers to the degree to which a successful solution process on a previous problem "fixes" a student's approach to a subsequent problem. Many students solve the following equations without seeing a connection between them.

Solve each of the following for x:

a. $2x = 12$

b. $2(x + 1) = 12$

c. $2(5x + 1) = 12$

The Hawaii Algebra Learning Project provides one example of an attempt to develop problem-solving processes within the teaching of the mathematics curriculum. This and other such efforts across grade levels may help to provide a model for solving problems that will stand the test of transfer beyond the world of the classroom.

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Discussion of Professor Rachlin's Paper

Sawada: Thank you Professor Rachlin for the interesting talk. Now let's have discussion.

Clarkson: I was struck with how often your students misread the problems and then later corrected that, but never seemed to consciously know that they had either read a sign wrong or read that something was squared or not squared. I was also wondering if their most common errors might have to do with misreading the problem and therefore they used the wrong operations or the wrong numbers?

Rachlin: Your observation is correct and the students do it a lot as we do when we are teaching. And one of the things that's hard to see is a videotape of your own teaching because of the number of statements you make that really aren't sentences. What happens is that the students frequently will be writing something correct and saying it incorrect, or they may do it the other way around. Part of it may have to do with the interview process, because you have got two cameras coming at you (one from the ceiling and one from straight on) and a microphone sitting by you and a professor sitting at your side waiting for you to make a mistake so that he can take it and show it to somebody and so on. Yet the students seem to be very comfortable as they are talking and like to talk about their own thinking. In most cases, they catch many of their own errors, a lot like what I do with my checkbook.

Another thing that happens with the students is that because this is a problem for them, their minds are working on the problem and they've turned off the mechanism that keeps checking their operations. So if you give the same problem to them in the standard textbook looking-fashion, they won't say that same piece because their mind is focused on the normal routine. One of the things that I would like to mention is, when you viewed the videotape, you saw the students doing the subtraction problem or addition problem vertically. Not only did we interview the students, but in the summer the teacher came in as well and we did the same tasks with the teacher that we did with the students. Then we asked the teacher, "How do you think Curtis is going to do on this problem? What do you think that Kathy is going to do when she solves the problem?" Next the teacher went and watched the videotape to see how Curtis did it and to see what Kathy did. Finally, the teacher came back and talked about it and said what surprised him and what didn't surprise him, to help us get a better understanding of what went on in the classroom. In doing that, the classroom teacher was very surprised at the vertical approach taken by three of the four students that we were interviewing. He was surprised because he had those students for three years now and he purposely avoided the vertical approach any time it showed up in the textbooks. He skipped over those problems. He didn't assign any. He knew they hadn't had that in class before and yet three of his four students, when they went to work these tasks, did them vertically. He chose not to do it because he felt it would be confusing to see both the vertical approach being used and a horizontal approach, and so he selected what he felt would be the best way to go for leading into equations as the horizontal approach. In his analysis, his reaction was that these are the top students in the school, so for them it's okay to have the two different solution approaches. If they had been a lower ability section, he still wouldn't want to have the students see both the vertical and horizontal ways to solve problems, he'd rather they have the best way.

Silver: I have a comment and then I would invite response from members of the Japanese delegation. The videotapes illustrated typical behavior of American students with respect to the use of the equality symbol, the equals sign. We noticed that in some of the students' work, especially one of the last pieces of work, the girl wrote down something like $4x^2+8x+2-2=0$. What she was doing is the $+ 2$ and $- 2 = 0$ without attention to the things had been written down earlier. Everything was written down on one line as if it were an equation, but in fact the equals sign was not being used properly in a mathematical sense, but rather as a signal of an intermediate result or some side calculation. That is one behavior that we notice very commonly with American students. The second thing is that at least one of the students remarked that whatever the quantity was on the right side of the equals sign was the answer. Even though the answer to the question that was asked was in fact something over on the left side of the equals sign, she referred to this quantity over here as the answer. That phenomenon of direction of the equals sign being from left to right is something else that has been observed with American and Canadian students and reported widely. I wonder if there are also typical characteristics of Japanese students or if there are differences?

Miwa: I will show two examples. One is in elementary school. Students often write $7 + 3 + 5 = 10 = 15$. The other is an equation in junior high school. Students often write

$$5x + 1 = 3x + 7$$
$$= 5x - 3x = 7 - 1$$
$$= 2x = 6.$$

This sort of error is common in Japan also.

Silver: As a follow-up in the explanation of the left to right pattern, the idea that what is on the right side is always the answer has been attributed to children's experience with arithmetic in which the computation is done before the equals sign is written and then the answer is expressed. Because Japanese schools have more experience with algebraic expressions earlier in their education than we do in the United States, would this be a similar phenomenon in Japanese schools?

Shimada: Yes, we also have a similar phenomenon. Usually Japanese elementary school teachers instruct their pupils to write an answer to a given problem in a format such as A.----(a Japanese character meaning answer instead of A, which comes from the English word 'answer'). In most cases, this is written in the right bottom of the given space. This practice makes clear what is considered as an answer.

In elementary schools, exercises are often given in a form such as $3 + 5 =$. In this case, the equal sign may be regarded as a command to do the given computation, not as a logical symbol for relation. Use of the equal sign as a symbol for relation, meaning that expressions on both sides being different names of the same things, comes at a later stage of the curriculum than its use as a sign for command. Many children seem to be unable to follow this shift of meaning. So to say, the first impression remains so strong that it may become uneasy to modify it by later instruction. Some educators call this phenomena the Effect of the First Impression.

Travers: I was dismayed but, unfortunately, not surprised by the embarrassment that the obviously bright student showed when she was solving that task and she said, "I'm sorry but I really didn't have the proper procedure to use." She was embarrassed because she didn't have a rule to follow. But I thought by what she did she showed a lot of understanding of mathematics and was using some fairly reasonable open-search procedures in her work. I also pick up from what little was said about the teachers that the students probably picked up this embarrassment because they felt that the teacher wouldn't approve of anything that didn't sort of follow a set of rules or procedures. Maybe that's unfair but that's what I picked up. But my comment is, I guess, that even with the very best students we just see very little evidence in these classes that there is much going on in the line of problem solving as we have been talking about this.

Rachlin: I'll comment on what you were saying, react to it. The teacher is viewed by the system as a very strong teacher, and also very sure of himself in that he is not only willing to come in for interviews with a professor, but he would come in long before we ever started interviewing any of the students just to talk. There's a difference of feeling about what the job of a teacher is that is reflected in what he is doing in the classroom, if it is teaching for problem solving or if it is teaching for being able to add, subtract, multiply, and divide.

Travers: And do well on examinations.

Rachlin: And do well on the exams. And he's been successful with that. None of the students were below the 80th percentile and they weren't supposed to be. But, even if you talk to him about problems that are word problems in the textbooks, his view of his job is to make it as easy for the students as possible, find the best way to go after them, teach that best way to your students and then they will be able to do well on those problems too. It's hard when I was interviewing not to try and help the student because you feel like they are suffering, but that is the teacher in us that was trying to help them by telling them something. One of the teachers that I had worked with a couple of weeks ago was saying, "You mean if I am going to teach problem solving that the students have to have some frustration?" Well, that might be part of it, but you have to start off with that, that's why you feel good after you solve the problem.

Wilson: Two comments. One in terms of your remark, Ken. I think there is evidence on these tapes of a lot of problem solving activity. The sad thing is they're made to feel guilty about showing it. My second remark is that there is an interesting paper floating around in unpublished form which will be published eventually. It is written by Alan Schoenfeld titled "When Good Teaching Leads to Bad Results." This paper reports cn a year of observing a really top notch teacher in geometry in New York state whose kids are successful on the Regent's Exam, on all those criteria, they are quiet in class and they work hard and they do all those things, and yet as Alan interprets this, they know very little about geometry at the end and they can't solve problems. That's available as a preprint only. It's not published yet, as far as I know.

Rachlin: I have one reaction to what you were saying. We talked for the last few days about the elementary school teacher and what their background and training was, and that the high school teacher has had a lot more mathematics. Yet I don't think it has improved the problem solving that goes on in their classrooms.

Wilson: I would challenge that there is nothing in the mathematics they have had that's problem solving.

Travers: To what extent has that teacher solved problems?

Nohda: In my opinion, in response to your comment about "When Good Teaching Leads to Bad Results," I think it depends on the level of students. For example, at the elementary school level, there good teaching does lead to good results; but on the high school level, if a teacher is too kind and explains too clearly to students, then they do not need to do any homework.

Wilson: I do believe very strongly that good teaching can lead to good results. That particular phrase was the title of someone else's paper describing the situation.

Miwa: I am very interested in your paper. I have one question. In your problems 1, 2, 3, and 4 in page 288, the term 'multiplied' is used. Multiplication is used in various ways in arithmetic and in algebra. In Japan, multiplication is introduced in the grade 2 of elementary school and there the meaning is clear--repeated addition. In later grades the meaning of multiplication is extended. My question is whether the student's understanding is better or not. Is the student's understanding in problem 1 and in problem 3 the same?

Rachlin: One of the segments that we cut off at the end was another student. The student was doing a multiplication of polynomials task, in fact it was problem #3 on that list. I'll tell you what the student did. As they started working on the problem, they weren't sure where to start and so they said, "I am going to try some numbers," which was a nice technique. When they used the numbers they used division to go ahead and solve the problem. So they went ahead and wrote this up as a division problem and started looking at it like long division of polynomials and then said, "I don't want to do this," and stopped. Then they wrote it like a fraction and wrote $2x^2 + 15x + 25$ divided by $x + 5$ and said, "I don't know what to do with this." They hadn't hit algebraic fractions yet. Then they looked at the problem for a while and paused and then said, "Oh, it's one of those bracket kind of problems," and went ahead and set up two sets of parentheses and then solved the problem. In that case her technique of multiplication has grown or could grow through that time, but it could also be that multiplication, even with numbers, has different meanings. We may only play one of the meanings for the students and not have them work with reverse operations as well as forward operations.

Hashimoto: Regarding research methodology, I think the method of thinking aloud is not common in Japan. I remember Professor Kantowski discussed the limitations of such an approach to research in a JRME article. Does anyone have suggestions of more effective or objective ways to studying this kind of situation?

Kantowski: I would like to respond to that. I think that there are weaknesses to the method of thinking aloud, but there are also very many strengths to this method. I think that we can learn much from having the students think aloud. Probably the ideal situation is to do some of each; to look at things objectively in some of the more traditional ways, and also observe students solving problems. The only way we can tell how a student arrived at the solution or how the student was thinking is to do some kind of clinical work and to observe some thinking aloud. Certainly there are weaknesses. The student may not be saying everything, but it is the best we can do. We don't have a better way of reading a student's mind.

Sawada: I'm sorry, but our time is now gone. Thank you.

Professor Travis' Paper

Ishida: Good morning. Now we would like to begin the morning session. The first speaker is Dr. Betty Travis. I will introduce her briefly. Her present position is Associate Professor in the Division of Mathematics, Computer and System Design, University of Texas at San Antonio. She is the principal investigator of several grants, including one from NSF on improving mathematics education and problem solving through the use of honors workshop for mathematics teachers.

PATTERN FINDING AS A HEURISTIC IN PROBLEM SOLVING: IMPLICATIONS FOR CLASSROOM TEACHERS

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In the past few years national organizations, commissions, and study groups have reported an evolution toward a mediocrity in public school education in the United States, leaving students ill-prepared to face the fierce competition of their future technological world (Note 1). These reports recommend strong action from all sectors of society to alter the course of this evolution (Note 2). Mathematics educators have also identified their own crises and concur that strong measures need to be taken to change the prevailing climate of routine testing-controlled curricula and externally-managed instruction to a climate of challenge and optimism controlled by well-educated and professional teachers.

In An Agenda for Action, the National Council of Teachers of Mathematics (NCTM) recommended that "problem solving be the focus of school mathematics in the 80's" (NCTM, 1980). This recommendation was based on the findings of an extensive survey of lay people and professional educators, funded by the National Science Foundation, called Priorities in School Mathematics. One of the barriers to implementation of the NCTM recommendations has been the question of how to teach problem solving and how to find time for it in the curriculum. The Conference Board of Mathematical Sciences in its report "The Mathematical Sciences Curriculum K-12: What Is Still Fundamental and What Is Not" recommended that the traditional mathematics curriculum of secondary schools be re-examined and streamlined to make room for the integration of problem solving and mathematical modeling techniques. While problem solving has been stated as the number one priority for school mathematics, little evidence is available or forthcoming on the commitment of classroom teachers to focus their class instruction on problem solving. No matter what actions are taken to address the issue of problem solving in the curriculum, the critical point will be the response our teachers make. It is not surprising that most reports conclude that teachers play the central role as agents of reform in the classroom. As stated at the conclusion of "Synthesis of Research on Improving Schools":

> Any improvement efforts in schools must begin with the concerns and needs of teachers; small steps toward improved practice are more important than any grand design. Teachers must be actively engaged in the improvement process. They must see the connection between what they are trying to do and what effects those attempts have on students. Finally, teachers must be recognized for the things they do well already and supported by people and resources for the new behaviors and procedures they decide to take on (Lieberman and Miller, 1981).

Teachers must be convinced that problem solving has a proper place in the curriculum. Teachers must commit class time in developing appropriate heuristics and incorporating them into all levels and topics of the curriculum. Teachers must believe that problem solving is a proper outcome of school instruction and that problem solving can be effectively evaluated along with other objectives. In essence, teachers must be convinced that problem solving is truly the number one priority and focus of school mathematics.

BACKGROUND

Since teachers are the "essential elements" of classroom instruction, their importance to the problem solving process cannot be ignored. Schoenfeld (1979) observed that students need explicit instruction in useful heuristics to develop a repertoire of strategies. For most students the most logical person to do this instruction is the classroom teacher. Therefore, it is imperative that leaders of mathematics education ensure that teachers are well-trained and well-experienced in problem solving processes.

This paper addresses the role of the teacher in the process and presents examples of teacher-training and inservices devoted to the general area of problem solving with special emphasis on pattern finding. These situations examine the teacher in the role of problem solver in pattern finding experiences.

PATTERN FINDING AS A HEURISTIC

Though problem solving is a complex human behavior and a personal, individualized type of activity that uses specific actions by students, Polya (1957) has identified general heuristics that are important elements in successful problem solving. One such heuristic is pattern finding. Lester (1980) defines the pattern finding approach as an organizer that facilitates the selection and use of a strategy. A cursory examination of the research literature in problem solving brought forth few studies that specifically address the issue of pattern finding as a heuristic since pattern finding does not fit the usual mode of problem solving research. The content variables describing the key elements of pattern finding do not necessarily fit into Krutetskii's broad subject areas of arithmetic, algebra, geometry, logic and general mathematics. But this brings an extra strength to the use of pattern finding as a strategy since the mathematical structure of the problem and the processes used to find the solution can draw upon any intersection or accumulation of knowledge and skills.

RESEARCH ON PATTERN FINDING AND RELATED HEURISTICS

Duncker (1945) introduced the concept of a "search model" in understanding the process of problem solving. A search model is a mental construct that provides the stimulus that begins and directs a student's actions. It also determines his or her perceptual field or "region of search." In the case of problem solving the region of search is the set of mathematical concepts and generalizations a student has learned. A student will then use this information to select appropriate goals or plans of procedures. Duncker believed it is the teacher's responsibility to help students conceptualize "search models" as they work through problems. Errors in problem solving occur when students use faulty, non-functional search models. One of the obvious differences between a student who solves problems readily and one who does not is that the better student has a more successful search model containing additional ideas on what might work, including pattern finding (Duncker, 1945).

Success with pattern finding necessarily involves the use of multiple heuristic procedures. McClintock (1979) identifies this as a key approach to solving problems. In addition to this, some heuristic processes may be embedded in others. Kantowski (1977) suggests that heuristic processes may be executed in some sequential pattern, such as the formation of a goal-oriented process, followed by analysis and synthesis. Therefore, we may not want to classify pattern finding as a heuristic in itself, but rather a procedure that suggests another, rather global, scheme.

There is some evidence of the Einstellung effect in pattern finding. Luchins (1942) defined the Einstellung effect as the type of mind-set that predisposes students to one type of thinking. Under this effect a student does not consider a problem on its own merits, but tries mechanically to use a previously successful method.

McClintock (1979) argued that even though a number of heuristics and strategies are involved in the problem solving process, the association of these processes with the task is dependent upon other variables. For example, the intrinsic problem structure can lead to different approaches by different problem solvers. Silver (1979) worked with the "pseudostructure" of problem types as a class of problems with similar attributes of the problem statement but not necessarily solvable by the same algorithm.

Polya and Kilpatrick (1974) suggested that the usual cases of pattern search underlie the induction process. Subsuming the induction process is the idea of generalization. Often induction involves searching a sequence of data to find a pattern. The pattern formalized provides the generalization. Since pattern finding is an element in induction and generalization, it is important that specific experiences designed to foster pattern finding abilities be provided in the curriculum. Techniques that use a categorization scheme of problem types to formulate solution stratgies can be explicitly given, as in cued induction procedures as identified by Schoenfeld (1979). In fact, looking for patterns is recognized as a cue for heuristic behavior--cues not in the sense that fundamental meanings and understanding of the problem situations are missing--but cues that provide students with more intelligent approaches to problem solving. Wilson (1967) studied the effectiveness of specific and general cues.

An advantage to teaching pattern finding strategies is that goal-oriented heuristic processes can be developed. Kantowski (1977) argues that goal orientation is important to problem solving, especially before regular applications of other heuristics. Of course, pattern finding can be taught for its own merit, though little has been done to isolate it as a strategy or study its effectiveness.

Some researchers believe that pattern finding may be inherent in all student strategies as they learn to solve problems. The Mathematical Problem Solving Project (LeBlanc and Kerr, 1977) observed over one hundred 4th & 5th grade students employing various strategies in problem solving during a 2-year period. Although students generally did not use any strategies at all, a few tried to identify patterns for some problems. Is pattern finding as much an intuitive process as one cued by the problem situation?

Webb (1979) has suggested that the solution of a problem involves not only the determination of the problem type from the statement, but also the particular algorithm or procedure used to solve the problem. Yet there is no algorithm for pattern finding, other than the general procedure of identifying the structure of the problem, describing the structure, and describing the pattern algorithmically.

PATTERN FINDING PROBLEM SITUATIONS

The National Science Foundation funded a series of Honor Workshops for mathematics teachers during a three-year period at the University of Texas at San Antonio. The focus of these workshops has been making better problem solvers of the teachers by simulating situations that allow not only "hands-on" experiences with mathematical ideas--but also "minds-on" experiences of using creative and novel strategies. It was important that specific experiences designed to foster problem solving abilities be provided so that mathematics teachers would be students of problem solving processes as well as students of mathematics.

NOTE: Any data presented have not been subjected to statistical analyses and are presented here only as descriptive information. These investigations were not formulated for hypothesis testing, but rather as exploratory searches to identify variables for further study.

Problem 1:

The following problem was given to elementary education mathematics specialists to illustrate the use of analysis and synthesis in a problem situation.

"What is the largest number of pieces of pie you can make with n cuts?" (Arithmetic Teacher, January, 1973)

When the problem was stated in this manner, few teachers could proceed with any solution. A few readily recognized an induction process and began to make a table with n=1, n=2, n=3, etc. If the problem was restated to ask for the maximum number of pieces of pie with 6 cuts, most students would automatically draw 6 lines (all intersecting in the center of the "circle"). If a new problem was written for the maximum number of pieces with 20 cuts, many students tried to do the actual slicing and counting and did not try table-making and inductive reasoning. If the problem was initially stated to cue the students to induction to answer for 20 cuts, students were able to solve the intermediate stages but were unable to generalize for n cuts.

Though many students saw the pattern, they were unable to generalize and conceptualize the pattern into an algorithm. Instead they expanded the table to 20 entries. No student was able to generalize the answer to n cuts without specific instruction in this process. Though the pattern was recognized, the algorithm was not obtainable. This example was useful as an illustration of analyzing a problem by determining the pattern and synthesizing the given information by incorporating the pattern into an algorithm.

Problem 2

The following problems were given to 45 high school mathematics teachers in the NSF-sponsored series on problem solving. Each student wrote out his or her thought processes and detailed his or her solution.

a) What is the last digit?

$$\underbrace{9 \cdot 9 \cdot 9 \cdot 9 \cdots 9}_{1000\ 9\text{'s}}$$

b) What are the last two digits?

$$7^{51}$$

c) What is the sum?

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{99\cdot 100}$$

It was assumed that the pattern developed in the parts a) and b) would cue the students as to the procedures to be used in the solution of the part c). But no student was able to find the sum. Samples of the strategies of part c) were:

"I tried looking for a pattern by adding 1/2 + 1/6, then 1/12 + 1/20, then. . . , but I could see no pattern emerging."

A few students had used the correct strategies but had observed no pattern because they had not reduced their answers. One student tried the more sophisticated mathematical approach of:

$$\sum_{n=1}^{99}\frac{1}{n(n+1)}$$

but was not able to find the sum.

Students were asked to re-work part c) with the following hints:

What are the sums?

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=?$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=?$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}=?$$

Most students used the following strategy (excerpted from one student's written account):

"I looked at the denominator and saw that it was 2*3 and I saw that the sum was 2/3. The next problem had the same pattern. The denominator was 3*4 and the sum was 3/4. So all I did was look at the denominator to give me the answer."

One person in a talk-through of his strategy stated that the rule was:

The sum of the first 2 terms is 2/3.
The sum of the first 3 terms is 3/4.
Therefore the sum of the first 99 terms is 99/100.

This general rule was to count the number of terms and state the rule as: the number of term/(number of term + 1). No other student related the sum to the number of terms in the addition process. Once the pattern of 1/[n*(n+1)] had been found, the algorithmic description of the process varied. This confirms the earlier statement by McClintock on the importance of the intrinsic structure of the problem leading to different approaches by different problem solvers.

Teachers expressed their concern that the problem in part c) in its original form was difficult because it involved the operations of both multiplication and addition. To investigate informally the assumption that the way a problem is presented and the type of information provided may significantly influence the success in solving it, this same problem was given in the format

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots + \frac{1}{9900} = ?$$

to another group of teachers. Instead of "simplifying" the problem so that there was only one operation to perform, this made the problem extremely difficult. The "pattern" became so deeply embedded in the structure of the problem that no student was able to uncover the correct algorithm.

Even though students conceptually understood that pattern finding was the only logical procedure to use in solving problems of this type, many failed to do so. For example, when given the problem in part c), many students did not recognize it as a pattern finding situation. Yet they indicated they "knew" to look for patterns in parts a) and b). When asked why this was so, every student replied "I knew you would not expect us to multiply 1000 9's or actually raise 7 to the 51st power." Yet they did not carry this same justification to part c). Students failed to recognize the critical element of thinking in problem solving exercises, especially within pattern finding, of replacing a problem goal with a series of easier, more readily solvable sub-goals.

Problem 3

In some problem solving situations, previous mathematics experience may actually hamper or interfere with a student's progress in solving problems. This makes it more difficult to isolate the key structures in problem solving and pattern finding.

To illustrate this interference effect for teachers, a nonmathematical "math" problem was given to several hundred pre-service and in-service middle school and high school mathematics teachers. (Find the rule: 8,5,4,9,1,7,6,3,2) Few have been able to find the "pattern," perhaps because the problem has usually been embedded in a set of exercises involving arithmetic or geometric sequences.

Perhaps this problem is not "fair" in the sense that it represents more of a "puzzle" than a problem situation. To some researchers puzzles of this type are not appropriate problems because the essential elements of understanding and meaning are lacking. The procedures used in solving puzzles or problems of this type are not usually transferable or generalizable (Brownell, 1942). But even puzzles can be useful tools to focus such thinking, because such experiences enlarge the student's repertoire of possibilities of strategies. It can also help the student analyze the task and organize his attack. It can also be useful to release the student from the inflexibility of a "mind set" that can block out all other possible avenues of solution. This is especially true when a group of pattern sequences have been given involving arithmetic or geometric progressions. It is difficult many times for students to leave that avenue of thinking and pursue other strategies.

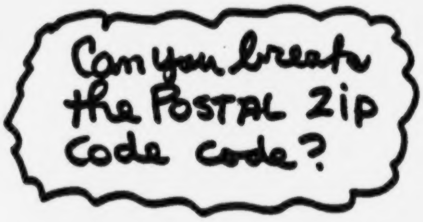
Problem 4

A problem that presented a challenge to the teachers was the following: (Problem presented by John Veltman, Mathematics Coordinator, NorthEast Independent School District, San Antonio, Texas.)

How does the code on the bottom of a postal envelope relate to the zip code? (See problem in figure 1.)

The most common approach was to try a binary formula to break the code. Pattern finding proved a far more effective procedure and provided successful experiences in pattern matching activities. Special problems like this can be given for special effects.

L D S, INC.
P.O. BOX 199
SAN ANTONIO, TX
78291-0199



Mathematics Coordinator
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10333 **Broadway**
San Antonio, TX 78217

0397495A SUP-EMC
JOHN VELTMAN
16585 BLANCO ROAD
APARTMENT 906
SAN ANTONIO TX 78232

Figure 1

American Association
for the Advancement of Science
1333 H Street, N.W.
Washington, D.C. 20005-9915

Diamond Shamrock
Refining and Marketing Company

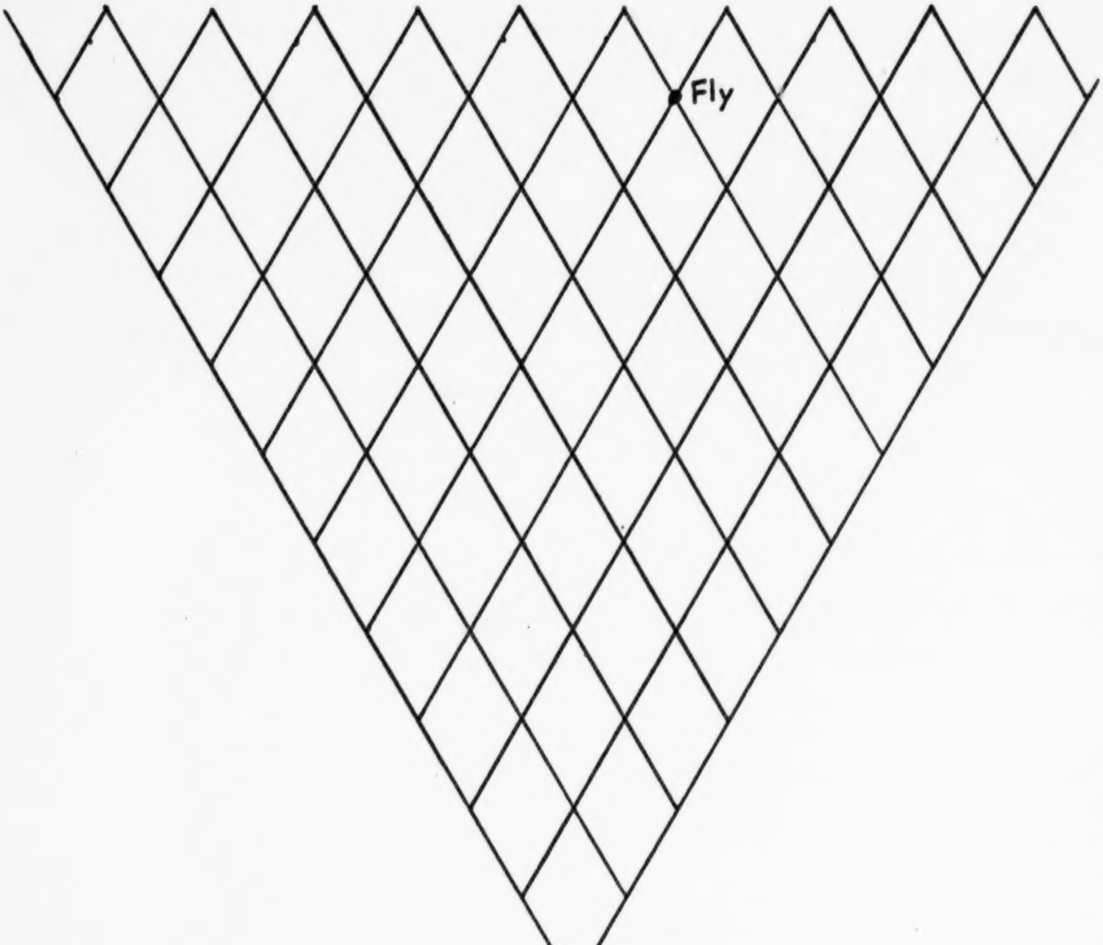
00004

POST OFFICE BOX 300
AMARILLO, TEXAS **79161-0001**

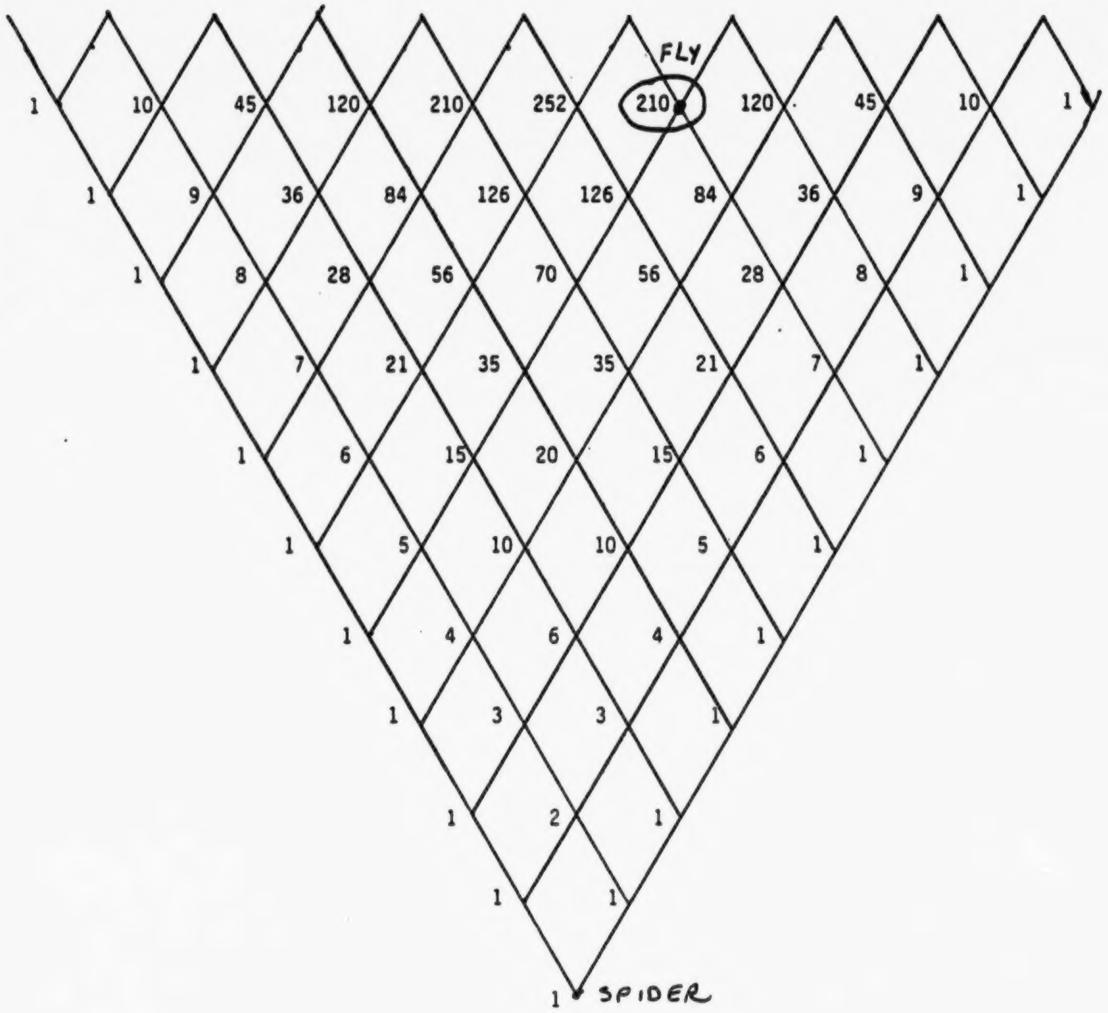
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1906 ASSOCIATION DRIVE
RESTON, VIRGINIA 22091-1593

Searching for patterns can lead to some exciting avenues of research within the experience level of all middle school and high school mathematics teachers. Edgell (1986) is constructing the expansion of $(1+x+x^2+...+x^m)^n$ by developing patterns within Pascal's triangle. For example:

Problem 5



How many ways can the spider get to the fly?


FLY
SPIDER

Integral Expansion of Pascal's Triangle Based Upon 2

R_1 1

R_2 1 1

R_3 1 2 1

R_4 1 3 3 1

R_5 1 4 6 4 1

R_6 1 5 10 10 5 1

R_7 1 6 15 20 15 6 1

R_8 1 7 21 35 35 21 7 1

R_9 1 8 28 56 70 56 28 8 1

.
.
.

$$\left(\frac{x^2 - 1}{x - 1}\right)^m = (x + 1)^m$$

$m=0$ 1

$m=1$ $1x + 1$

$m=2$ $1x^2 + 2x + 1$

$m=3$ $1x^3 + 3x^2 + 3x + 1$

$m=4$ $1x^4 + 4x^3 + 6x^2 + 4x + 1$

$m=5$ $1x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

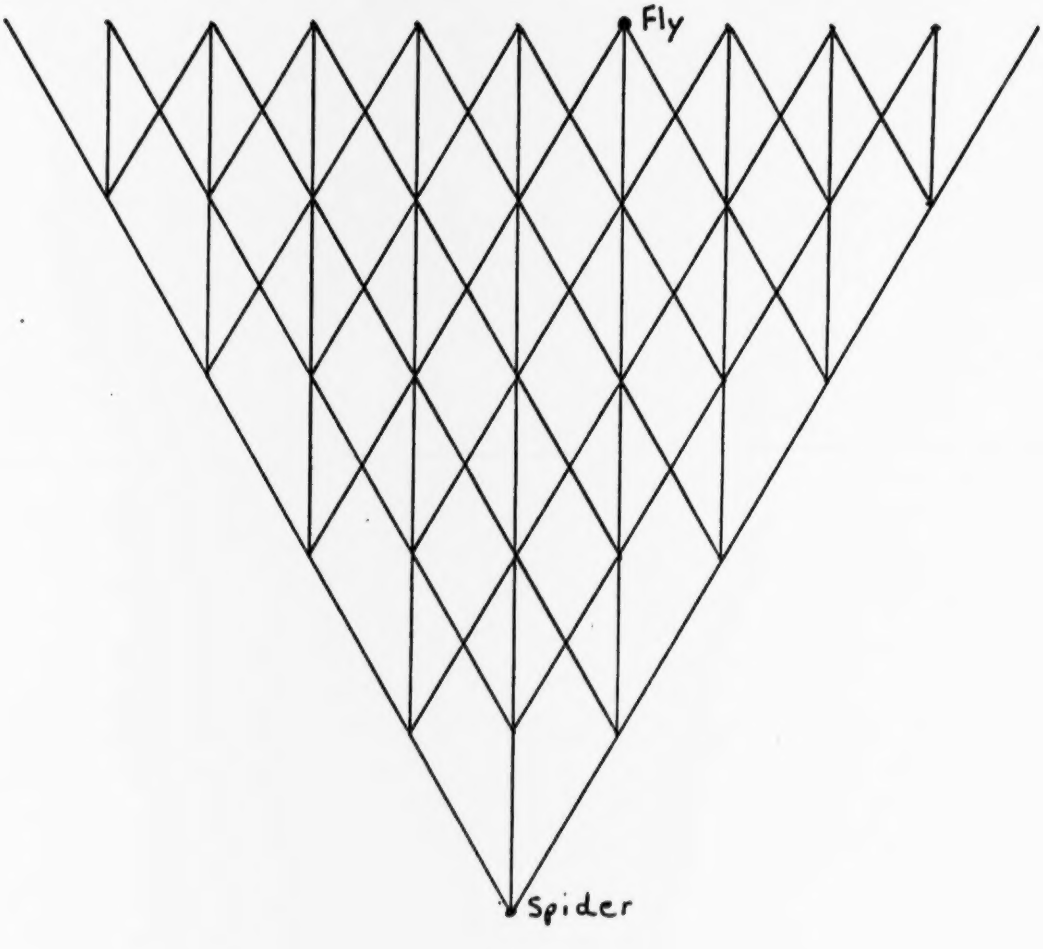
$m=6$ $1x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$

$m=7$ $1x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$

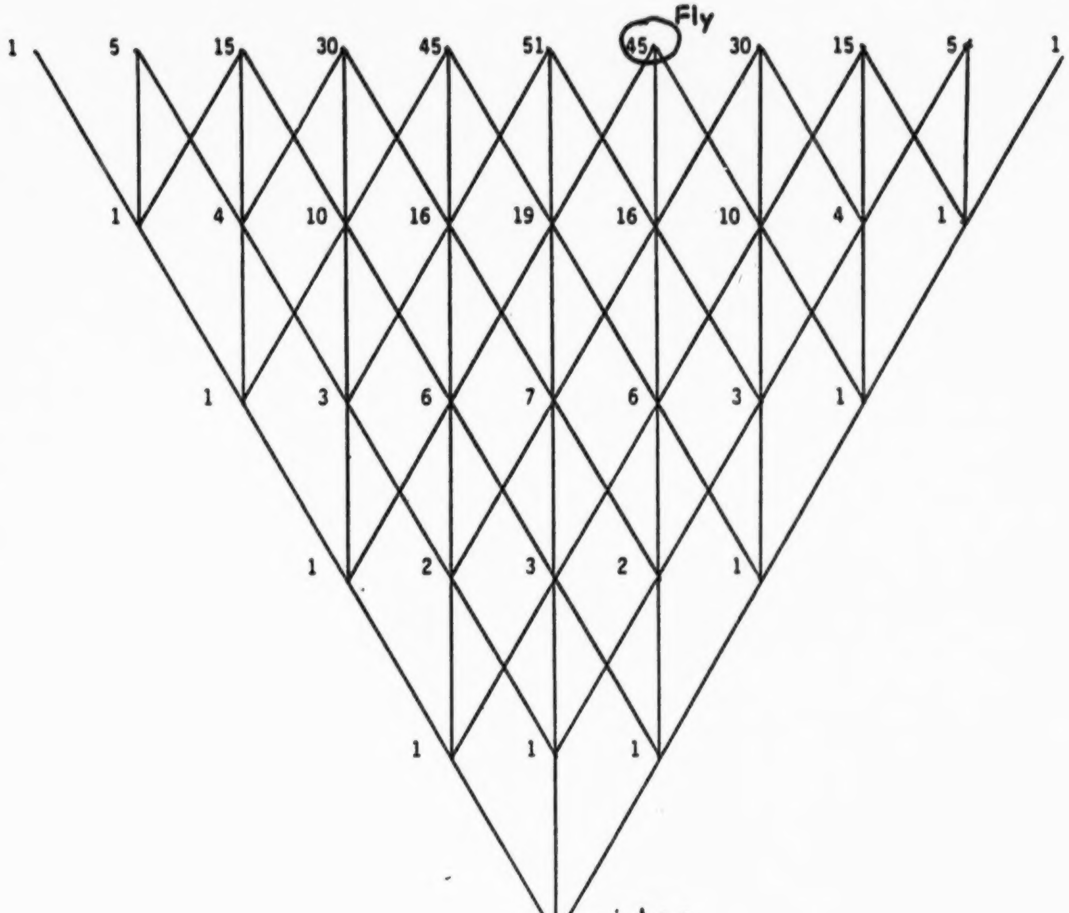
.
.
.

R_{m+1} Corresponds to the coefficients of $(x+1)^m$

Now how many ways can the spider get to the fly?



Each element in a row is the sum of the three numbers below it in the table.


Fly
1 5 15 30 45 51 45 30 15 5 1
1 4 10 16 19 16 10 4 1
1 3 6 7 6 3 1
1 2 3 2 1
1 1 1
1 Spider

Integral Expansion of Pascal's Triangle Based Upon 3

R_1						1					
R_2					1	1	1				
R_3				1	2	3	2	1			
R_4			1	3	6	7	6	3	1		
R_5		1	4	10	16	19	16	10	4	1	
R_6	1	5	15	30	45	51	45	30	15	5	1
⋮						⋮					

$$\left(\frac{x^3 - 1}{x - 1}\right)^m = \left(x^2 + x + 1\right)^m$$

m=0 1

m=1 $1x^2 + 1x + 1$

m=2 $1x^4 + 2x^3 + 3x^2 + 2x + 1$

m=3 $1x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$

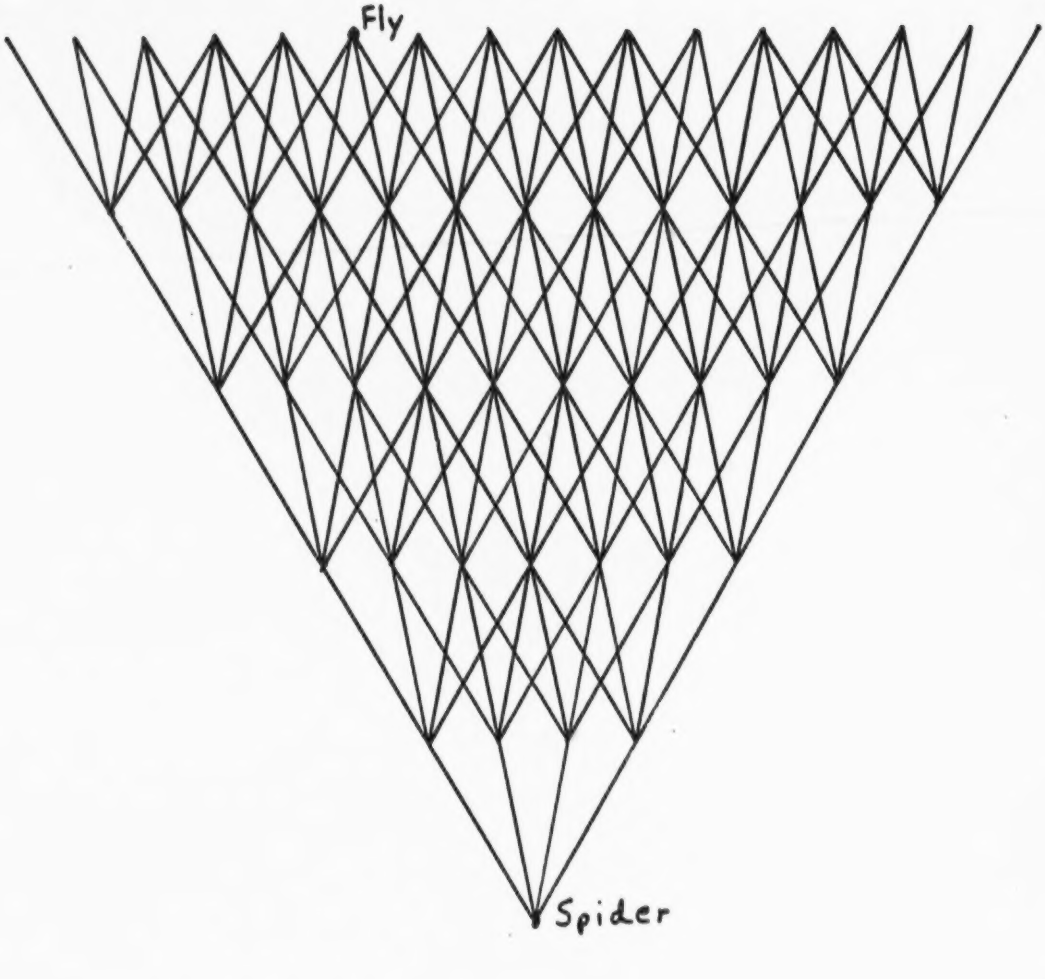
m=4 $1x^8 + 4x^7 + 10x^6 + 16x^5 + 19x^4 + 16x^3 + 10x^2 + 4x + 1$

m=5 $1x^{10} + 5x^9 + 15x^8 + 30x^7 + 45x^6 + 51x^5 + 45x^4 + 30x^3 + 15x^2 + 5x + 1$

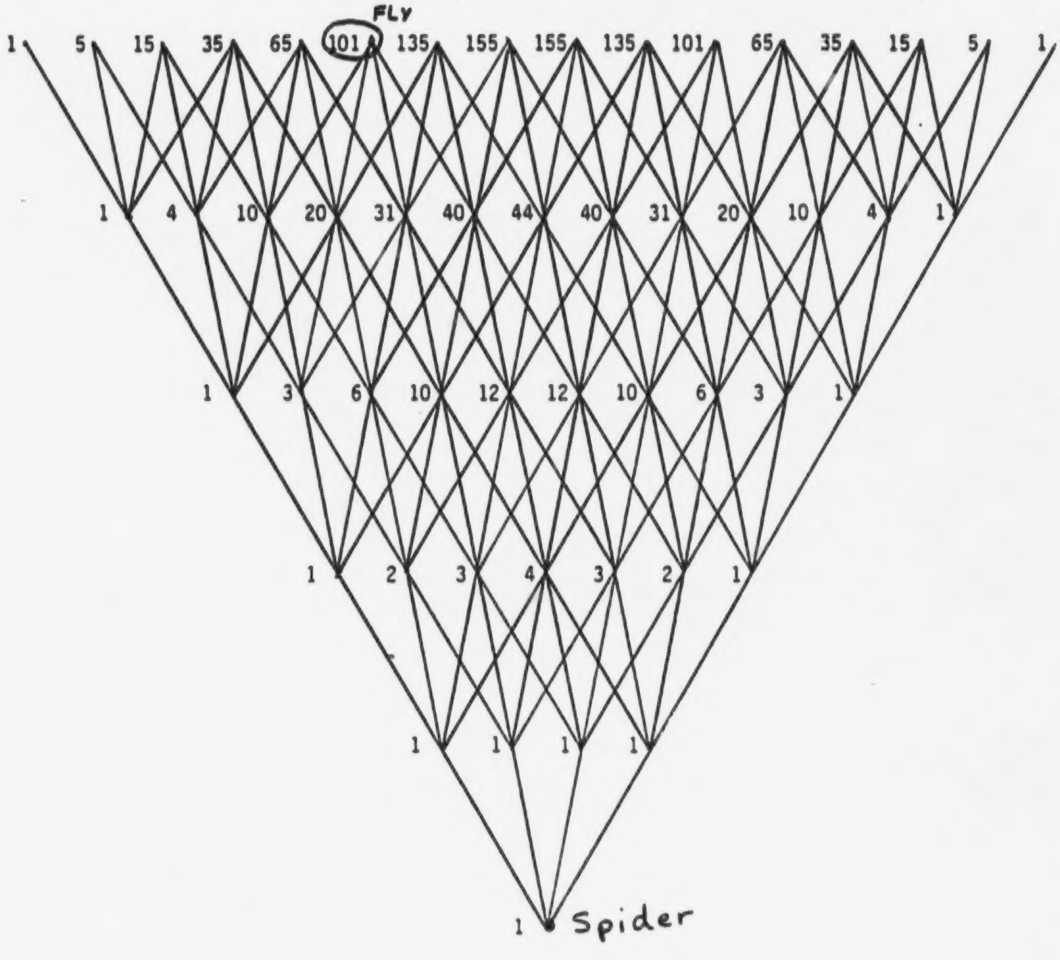
⋮ ⋮

R_{m+1} Corresponds to the coefficients of $(x^2+x+1)^m$

How many ways can the spider get to the fly?



Each element in a row is the sum of the four numbers below it in the table.


FLY
1 5 15 35 65 101 135 155 155 135 101 65 35 15 5 1
1 4 10 20 31 40 44 40 31 20 10 4 1
1 3 6 10 12 12 10 6 3 1
1 2 3 4 3 2 1
1 1 1 1
1 Spider

Integral Extension of Pascal's Triangle Based Upon 4

R_1 1

R_2 1 1 1 1

R_3 1 2 3 4 3 2 1

R_4 1 3 6 10 12 12 10 6 3 1

R_5 1 4 10 20 31 40 44 40 31 20 10 4 1

R_6 1 5 15 35 65 101 135 155 155 135 101 65 35 15 5 1

$\vdots$ $\vdots$

$$\left(\frac{x^4 - 1}{x - 1}\right)^m = \left(x^3+x^2+x+1\right)^m$$

m=0 1

m=1 $1x^3 + 1x^2 + 1x + 1$

m=2 $1x^6+2x^5+3x^4+4x^3+3x^2+2x+1$

m=3 $1x^9+3x^8+6x^7+10x^6+12x^5+12x^4+10x^3+6x^2+3x+1$

m=4 $1x^{12}+4x^{11}+10x^{10}+20x^9+21x^8+40x^7+44x^6+40x^5+31x^4+20x^3+10x^2+4x+1$

m=5
$1x^{15}+5x^{14}+15x^{13}+35x^{12}+65x^{11}+101x^{10}+135x^9+155x^8+155x^7+135x^6+101x^5+64x^4+\ldots$

$\vdots$ $\vdots$

R_{m+1} Corresponds to the coefficients of $(x^3+x^2+x+1)^m$

Problem 6

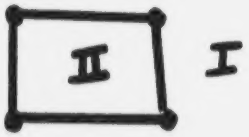
A key element in pattern finding is using simpler cases and making a table. But a more difficult aspect of this procedure is to present a problem with several variables and try to isolate each one by the use of several tables. For example, in trying to have teachers "discover" Euler's formula for two dimensions

$$V + R - C = 2$$

where V is the number of vertices, R is the number of regions, and C is the number of curves, several tables are constructed so that different variables can be isolated.

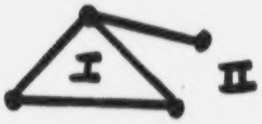
Teachers are given the following illustrations of vertices, regions and curves.

eg.



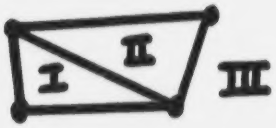
4 vertices
2 regions
4 curves

eg.



4 vertices
2 regions
4 curves

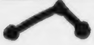
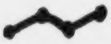
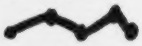
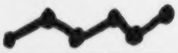
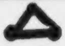
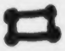
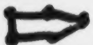
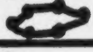
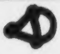
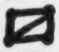
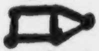
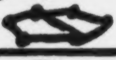
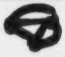
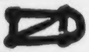
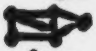
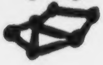
eg.



4 vertices
3 regions
5 curves

Problem: Find a relationship between the number of vertices and curves in a network and the number of regions it separates in the plane.

Fill in the following table:

V	C	R	
3	2	1	
4	3	1	
5	4	1	
6	5	1	
3	3	2	
4	4	2	
5	5	2	
6	6	2	
3	4	3	
4	5	3	
5	6	3	
6	7	3	
3	5	4	
4	6	4	
5	7	4	
6	8	4	

R is kept constant so that V can vary; teachers then discover the effect on C. V-C is constant and describes a linear function,

$$F(R) = V - C$$

By observation $F(1) = 1$, $F(2) = 0$, $F(3) = -1$, etc. By graphing these values, teachers discover that $F(R) = 2 - R$. Therefore,

$$V - C = 2 - R, \text{ or}$$

$$V + R - C = 2$$

Problem 7

Computers provide additional opportunities for exploration into pattern finding experiences. Schwartzman (1986) used a computer to investigate the factorability of quadratics with consecutive integer coefficients. For example,

$$232\ x^2 + 233\ x - 234$$

is factorable as: $(29\ x - 18)\ (8\ x + 13)$.

By studying the patterns of coefficients and recalling the form of Fibonacci and Lucas numbers, general algorithms can be developed as "rules" of factorability of quadratics in the following form:

$$n \cdot x^2 + (n + 1) \cdot x - (n + 2) = 0, \ (n \geq 1)$$

It was difficult for teachers to determine on their own the role of the Fibonacci and Lucas numbers from the tables without directed instruction. This problem became a quick favorite with teachers since it allowed them to create quadratics of the special form and use the computer to test out possibilities for factors from the Fibonacci and Lucas numbers.

Another problem utilizing the computer in pattern finding involves the prime factorization of a number. The computer can be used to calculate $\tau(n)$ [the number of positive divisors of n], $\phi(n)$ [the number of positive integers less than or equal to n which are relatively prime to n], and $\sigma(n)$ [the sum of the positive divisors of n]. Tables can be constructed and patterns can be uncovered. For example, teachers were able to "discover" that a perfect number is one in which $\tau(n) = 2n$. Other "discoveries" included the fact that $\tau(n)$, $\phi(n)$, and $\sigma(n)$ are multiplicative functions such that (if a and b are relatively prime):

$$\tau(a * b) = \tau(a) * \tau(b)$$

$$\sigma(a * b) = \sigma(a) * \sigma(b)$$

and

$$\phi(a * b) = \phi(a) * \phi(b)$$

Extensions to this problem provided additional "discoveries" and conjectures.

ISSUES FOR CONSIDERATION

Problem solving must be taught through the design of carefully developed sequences of activities and appropriately organized experiences. By recognizing that certain processes best fit certain problem categories, teachers are able to select appropriate tasks and set instruction to the teaching of these tasks for the purpose of improving problem solving.

Instructional processes can facilitate pattern finding but these must also be planned. Future efforts need to clarify pattern finding as a specific heuristic and determine its position within a hierarchy of multiple heuristics. Most studies can recognize those problems that make use of pattern finding as an integral part of the solutions, but there has been no systematic development of instruction and no taxonomy of classification of problem types.

MORE GENERAL ISSUES

Beyond the more specific issue of pattern finding as a heuristic in the problem solving process is the more general question of problem solving and its place in the curriculum. This paper began by stressing the role of the teacher in the process of problem solving. This paper ends with a call for a commitment to the following needs in public school mathematics teaching for the improvement of problem solving.

(1) There is the need to address the concern of making time for problem solving, including pattern finding, in the curriculum, especially within the confines of detailed curricula being implemented as mandatory for all schools in some states. This need is real and immediate since most teachers feel there is simply no time to diverge from the book's content and sequence of topics if one is to cover the course objectives.

(2) There is the need to train teachers in the art of teaching pattern finding and other heuristics and integrating it into their regular classroom practices.

(3) There is the need to inform teachers how to make use of other resources such as computers and calculators to enhance regular classroom instruction and problem solving strategies.

(4) There is the need to focus on teaching of problem solving from the perspective that it can be incorporated into each topic of the curriculum and effectively evaluated along with other objectives. Many teachers hold the attitude that problem solving constitutes a separate topic that must be taught apart from other topics.

Research in problem solving attempts to clarify complex issues and provide teachers professional assistance with classroom decisions. It is our hope that the time will soon come when all teachers have the competence and confidence to make professional judgments that capitalize on the products of research. Perhaps then teachers will understand the relationship between concepts and skills and will devote more classroom attention to problem solving in general and pattern finding and other heuristics in particular.

Notes

1. "An Open Letter To The American People--A Nation At Risk: The Imperative For Educational Reform." Final Report of the National Commission on Excellence in Education, April, 1983.

2. "Action For Excellence: A Comprehensive Plan To Improve Our Nation's Schools." Report of the Task Force on Education for Economic Growth, June, 1983.

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Discussion of Professor Travis' Paper

Ishida: Thank you very much for your very interesting presentation. Now, we would like to discuss the presentation. Are there any questions or comments?

Hashimoto: I have one question. Regarding the first idea on more general issues on page 330, I was surprised that in the United States most teachers feel there is no time to diverge from the book's contents and so on. I can understand such a situation in Japan, but I thought it was different in your country. Could you give more information about this matter?

Becker: I think I understood at least part of what Professor Hashimoto was asking. Professor Travis is saying that we need to make more time for problem solving available, and Professor Hashimoto, correct me if I am wrong, is saying that he thought that in the United States we have a lot more freedom to do this. In some respects I think that's true. I believe that teachers, if they decide they want to do some things that they feel are important, there is of course some freedom to do that. But though we don't have a national syllabus, nor even a state syllabus, still there is a lot of uniformity in terms of mathematical content that has to be taught in the classroom. It may be dictated in part, for example, by various standardized tests that will be administered in the school. I think there is a big problem, a very important one, that Professor Travis raises because if we were to put more emphasis on problem solving activities in the classroom, it will require a lot more time. We need to study that problem.

Shimada: In order to have some time for problem solving, we must cut out something from the existing curriculum. If we emphasize this aspect, what should be discarded?

Travis: I think there are several ways that you can make time in the curriculum. Let me mention, as a response to an earlier question before I answer you specifically, that I come from a state where the curriculum is mandated and it's very basic skills-oriented and our students are tested using a state test. They take tests in grades 1, 3, 5, 7, 9, 11 and then a graduation test, and the tests all concentrate on basic skills. Teachers know they must teach to those tests, they must document the fact that they are actively teaching what are called essential elements of mathematics. They have to document reteaching, there is this constant pressure to cover a list of very basic skills. What we are trying to do is to help them prioritize the topics to say, yes, you must cover those topics, but you can prioritize them and you can combine them. You can make use of the computer also to demonstrate certain topics so that less time will be spent on others and more time can be spent on things that are perhaps more important.

Inouye: Are you trying to convince legislators or the teachers?

Travis: The teachers. If any changes are to be made in school mathematics, those changes must come from the teachers.

Shimada: In order to realize such ideas, I think the present system of teaching basic skills, so to say, such as concept of number and related ways of computation, should be all reorganized under the consideration of the use of computer and its influence on society. I doubt that the developmental stages of children's numeracy are the same as those we are now assuming in constructing its existing curriculum. Children will live in a quite different culture and social environment from those in which we were brought up. They will be surrounded by many computerized systems. In that world, children's psychological development of number concepts or any other matters may differ from those of previous times. So deeper research is needed in this aspect, through which we may find a way to make the time for basic skills shorter and have more time for problem solving. To realize this, a study of the curriculum is also necessary.

Inouye: What kind of use of computer do you expect in classroom?

Shimada: There may be many uses. But what I have just said is not directly related to any specific use of computer in the classroom. It is that the society children live in has changed and stimuli from it and needs to adapt to it are different from yesterday, perhaps in a positive direction so far as numeracy is concerned; i.e., less need for paper-pencil computation and more need for principles and appreciation for many uses of numbers.

The important thing is to learn the principles. I think we agree with that.

Kantowski: I would like to respond to the notion that teachers say that they have no time to cover the amount of material that needs to be covered and to add problem solving on top of that. I think that there is a lot of inefficient use of time in the classroom. As Dr. Wilson mentioned the other day, when teachers participate in intensive workshops and are totally convinced of the value of all of this problem solving instruction, they go back to their classrooms and they somehow find the time to do the problem solving in addition to the other things they are doing. I think this is saying that teachers (1) need to be so convinced of the importance of problem solving, that they begin to see where it can fit into the curriculum and (2) in doing this, they get a different point of view of the curriculum and they develop what Dr. Wilson calls ownership of what they are doing in the classroom. As a result they begin to be able to make decisions about what are the important things to spend a lot of time on and what are some of the things that they can spend less time on. In general, the experience has been that the teachers who do spend time on problem solving in the classroom have students who do not suffer on the examinations. Their students do very well, it is just that they themselves reorganized what they do in the classroom. So that I agree with Dr. Travis that the important thing is getting to the teachers, getting them convinced of the importance of this, and getting them to feel an ownership of what they do, and getting them to make the decisions about how to deal with their time in the classroom more efficiently.

Rachlin: What I am not sure of is whether there is actually evidence that says that teachers who have gone through workshops like these do teach differently in their classrooms, that it has been measured, and they do it over a period of time differently, not just the year after they finished the workshop.

Travis: I would like to just follow-up to Dr. Kantowski. After one of our formal workshops, we did a follow-up on an algebra teacher. She was very resistant to putting forth any kind of problem solving effort in her classroom in the beginning, but Dr. Burke, who was co-investigator with me on the grant, worked with her during the fall semester. After about two months of intensive in-class problem solving instruction, she was about two weeks behind other algebra teachers. But by the end of the semester she had caught up and, in fact, her students were doing much better on problem solving and other activities than those in the other classrooms. She became convinced and gave practically a testimonial at the convocation that we had that winter to the other teachers. Again, that is just one experience. We still see her in the summers and we know she is still participating and emphasizing problem solving, but that may be because of our diligent follow-up and work we do with her each summer.

Sugiyama: In Japan, it is common to use pattern finding. It is sort of a standard procedure and I wonder whether pattern finding is common in the U.S.?

Travis: Perhaps some teachers use pattern finding though it's not an integrative part of most curricula. What I am trying to stress to the teachers is that pattern finding needs to be part of a guided instruction--that it needs to be a planned effort for the student. But at least within our state there is no set curriculum devoted to pattern finding. Again, you may find a little instance of it here by an individual teacher or in a book or something like that, but there is no program.

Nohda: In Japan, in the early years teachers use problem solving strategies to teach basic contents and this requires longer hours, but in later years the advantages are seen. There are teachers who use problem solving in basic contents and there are many reports which show that they obtain good results because of using problem solving techniques early. They make effort to continue to use problem solving in order to obtain good results.

Hashimoto: In Japan, the textbooks include problems in which students make a drawing or a diagram. Problem solving is incorporated in this way too.

Sugiyama: So we incorporate into the textbooks problems that do use problem solving techniques, much more so than in American schools.

Clarkson: I will just point out that now we are putting more emphasis on problem solving and pattern finding. This is starting in very early grades. However, there is not as much of an emphasis on making a generalization. So, where they are naming a rule, many times the rule is to find how to get from one element to the next. If there is any generalization to be made, it is usually a quite simple one. However, there is a lot of pattern finding in the textbooks, especially the new ones.

Travis: Also, I would like to state that just because it is in the textbooks we still need to convince the teachers to incorporate it into their planned instruction. In our state, if it is not one of the essential elements, the students are not going to be tested on it and it will not be taught. Once again, I really think the core of the problem will be with our teachers, the response our teachers make to our plans.

Rachlin: We are talking again now about elementary school texts. Are you also referring to high school texts in terms of looking at pattern finding, and are there examples of that that occur in the high school books and in high school classes in Japan?

Nohda: In the lower grades there is time so what we have been saying about using problem solving techniques in basic education is generally true, but as you go up to the upper grades, especially in high school, the teachers don't have time, they have to prepare for exams.

Shimada: I will comment on Japanese comments. Surely many Japanese elementary school textbooks use pattern finding approaches, for example, by making a table to introduce new concepts. However, in table-making, a tabular form with blank spaces is provided, names of variables are given in the table, and values of the independent variable are also given. Values of the independent variable are arranged in a systematic way in the row of that variable (sometimes a few being left blank). Children are required to fill in the blank spaces and afterwards to find a certain pattern in arrangement of numbers. By this method, maybe the chances to consider important stages in pattern finding such as identifying appropriate variables and arranging possible values of the independent variable in a systematic way are missed. All of the above are already given in the text, there is no need to consider anything! Another important step may be the isolation of variables, but I will not spend more time on it now. Thus, though Japanese textbooks use a table-making approach or pattern-finding approach, they are not used intentionally to teach the strategy, but rather as a technique to introduce a new concept.

When I observed an experimental lesson in which children were busy working on gathering data for a physical experiment to find out a relation between data, I noticed that children did not hit upon arranging data in a systematic way, i.e., increasing or decreasing order of one variable, and could not succeed in finding out a relation. Suggesting to students that they arrange their data in a systematic way would have helped them to discover a relation. We are likely to assume that such thinking is quite natural and need not be suggested, but such is not the case.

Sugiyama: There seems to be a difference of opinion. I gather you understood that. It is true that students use tables, for example, as a problem solving technique, but the table is already there and all the student does is to fill it in, right? Well, what I am saying is that it is true that they started that way, but in solving word problems later on, the kids do the thing themselves and they organize the table themselves.

We can't make general statements for all teachers naturally. Making tables is common in elementary school. But it gets less common from junior high school on.

Wilson: Before the break, Betty, I think you should give us the solution to the sequence.

Clarkson: Good, but one thing I thought is that the problem is not translatable to the Japanese, so unless they work the problem in English they would never be able to solve it.

Travis: I will give a response to that, and I would also like to end with the idea that there are definitely some advantages of having pattern finding within a curriculum. For many students, the problems are motivating. The nonroutine situations are motivating. We find also that at least in the American schools the reading level is a difficulty for many students. You can give pattern finding type of problems that students can still work and read and yet will still appeal to them and their interests.

Now to the solution of the problem I gave you yesterday. Everyone I have given the problem to with a mathematical background approaches it in some type of sequence and tries to find a pattern to the sequence. Again, I realize it is not really fair here, but it is just to talk about the interference effect sometimes that our background can bring to problem situations, and that's its purpose--a special problem given for a special purpose - not generalizable in other kinds of situations.

The pattern is E F F N O S S T T--the digits listed in alphabetical order.

Professor Nohda's Paper

Clarkson: Our next speaker is Professor Nobuhiko Nohda. Professor Nohda is a well-known researcher in mathematics education from Tsukuba University. He has done extensive research on open-ended problem solving and on pattern finding in problem solving. Professor Nohda has given many papers at scientific meetings not only in Japan, but also at various international conferences. Professor Nohda, we look forward to your talk.

MATHEMATICAL PATTERN-FINDING IN PROBLEM SOLVING--
MODE AND PERFORMANCE OF JAPANESE STUDENTS

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Purpose and Procedures

The teaching of school mathematics in Japan has been widely recognized as an opportunity for students to learn about problem solving (Nohda, 1983). Recently Japanese teachers have been working out a teaching plan for problem solving (Nohda, 1986). As a problem is able to be taken in a number of different senses (Lester, 1982), I will describe the problem as follows: A problem occurs when students are confronted with a task which is usually given by the teacher. Often there is no prescribed way of solving the problem. Students are able to solve the problem when finding a suitable "PATTERN" in the problem. On the other hand, they would have some difficulties in solving their problems when not being able to find a suitable "PATTERN." To study the strategies of problem solving and the difficult points of them is to make it clear how students find suitable patterns for the problems, what strategies they apply in their problem solving, and in what parts they have difficulty (Silver, 1985).

For the purpose of this study, first of all, I consider the following two cases. One of them is the underlying pattern in the problem; that is, the nature of characterizing the problem itself. The other is the feature of strategies in students' problem solving. The former means the structure of the problem, the rule in it and so forth. The latter is the mode of action applied in students' problem solving. Therefore, in order that students might do better in their problem solving, it is necessary for these features to be more consistent. In the case of the students who fail to solve problems or feel it difficult to do so, the reason would be that there is a "gap" between such two features (Mayer, Sowder, 1985).

To make clear the student's strategies and difficulties in problem solving from the two viewpoints as above, I will present a more difficult problem than the problem found in the normal textbook (Christansen & Walter, 1986). Then, I will observe the mode of action on students' problem solving and analyze the process of problem solving which students take a step to solve the problem, and whether the students arrive at the correct final answer or not. First, I will describe the pattern of the problem as mentioned above and secondly consider the mode of student's action. From these two aspects, the strategies and the difficult parts in student's problem solving will be made clear.

Subjects in this study are the elementary school students who are living in Tsukuba New Town near The University of Tsukuba. There are three problems: problem 1 for 2nd graders and problems 2 and 3 for 5th graders. Problems 1 and 3 are concerned with counting the number of figures, while problem 2 is concerned with computation.

The Case of 2nd grade-students of the elementary school

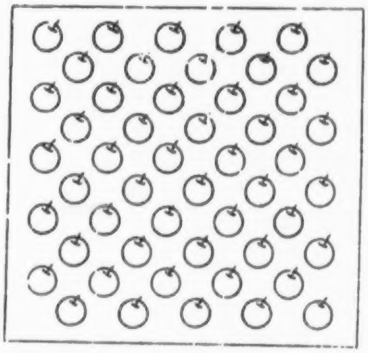
PROBLEM 1 (See Fig. 1)

(1) How many apples are there in this figure?
*Count the number without any skipping or overlapping.

(2) What way of counting and how many ways of counting do you think of?

(3) Of all of your ways of counting, encircle the one you think the most suitable.

Fig. 1 Apple Problem



The way of the research is that the students are given a picture of "apples" and asked to write the answers on separate sheets, after the teacher reads the problem.

The intention of this problem lies in that at a first glance the students feel it rather difficult to count well because of seemingly complicated configuration. For 2nd-grade students, it is said that it is difficult for them to count well after arranging and grouping objects into sets of equal number or by the concept of one-to-one correspondence, although it is easy for most students to count up to 50.

The feature of the pattern of the problem is that students feel it's difficult to count correctly. After observing in details, however, they will notice the following features.

One is to take the apples as two sets each forming 5 columns x 5 rows. That is, there are two sets of 5x5, so the pattern here is 5x5x2. The apples arranged by 5x5 can be taken as such and not be confused by slight shifting. Therefore, the key points of the pattern finding in this problem can be asserted to be finding these features (See Figure 10).

From another point of view, it can be taken that they are arranged in 10 horizontal lines, each having 5 apples. In this case, it can be taken 5x10. In reverse, taken 10x5.

On the other hand, the following two features can be seen by diagonals (See Figures 8 and 9). One feature is that they are symmetrical with AC as a diagonal, 2+4+6+8+10+8+6+4+2 as the way of arranging, and (2+8)+(4+6)+10+(8+2)+(6+4) as the pattern of counting. Another is the following one with diagonal BD.

(1+3+5+7+9)+(9+7+5+3+1)=(1+9)+(3+7)+5+5+(7+3)+(9+1)

Then, how do they actually find the patterns of this problem? The mode of action considered from students' scripts will be examined.

Students solved the problem in different ways as follows:

(1) Counting apples one by one or with the number filled in the sketch of each apple. Some students made counting mistakes in their processes. (See Figure 2)

(2) Counting apples in pairs. Some students who counted the number of pairs encountered the difficulty of calculating 2x25 and failed. (See Figure 3)

Fig. 2 Counting 1 by 1

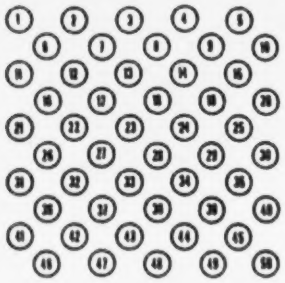
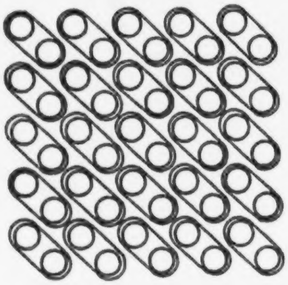


Fig. 3 Counting in Pairs



(3) Counting apples in groups of five. (See Figures 4 and 5) Students who counted five apples together counted relatively quickly and accurately. Some students counted in horizontal lines. (See Figure 5)

Fig. 4 Counting 1 to every 5

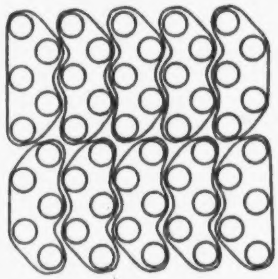
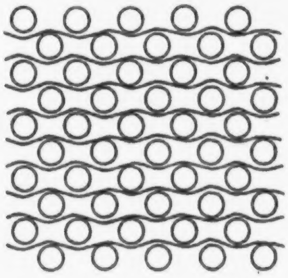


Fig. 5 Counting 1 to every 5



(4) Counting apples in groups of ten. Although this idea is good, in the case of Figure 6, a few of the students failed to make groups of ten and in the case of Figure 7, there were no mistakes.

Fig. 6 Counting 1 to every 10 (Grouping)

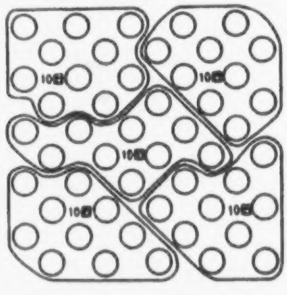
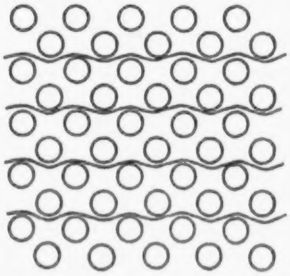


Fig. 7 Counting 1 to every 10 (Horizontally)



(5) Counting apples through consideration of the geometrical configuration.

A rather small number of the students used Figures 8 or 9 as the way of counting. In this case, adding numbers aslant was the key, but most students seemed to reject it. Most students seemed to feel it troublesome to count the numbers and then calculate.

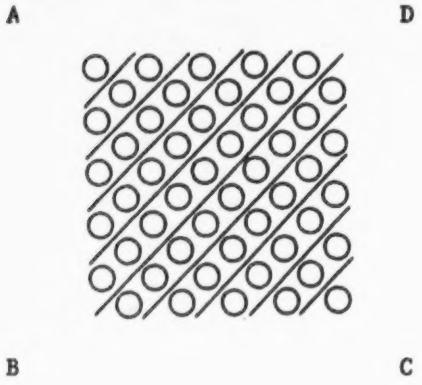


Fig. 8 Counting Along Aslant

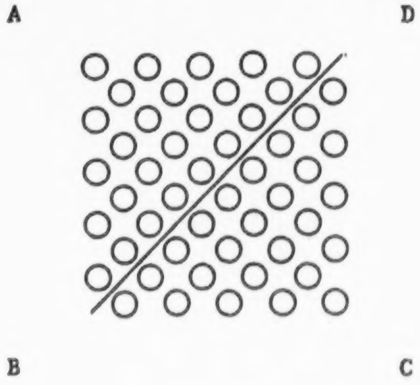


Fig. 9 Counting Divided by Diagonal

On the contrary, in the case of Figure 9, after counting the numbers along the slanting line (position), students got 50, because two groups calculating 1+3+5+7+9 which led to 25 were taken. Students in this process, however, did not conceive that 1+3+5+7+9 could be calculated by adding 1+9, 3+7, and 5.

Next, most students chose the cases of Figures 5 or 7 as the best way for the question of which way to count the apples, after explaining on the blackboard.

By the way, to summarize the students' responses, students seemed to take the strategy of counting in groups of the length-width formation and the slanting formation. Students found the same number and rule (pattern) in groups of such formations as the case of Figure 10.

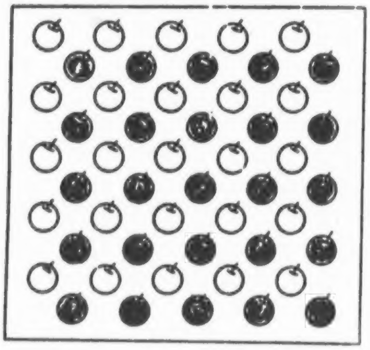


Fig. 10 Gestalt of Apple Problem

The ways of these countings were related to each other. Concretely, Figures 3, 4 and 6 were related in that they were grouped according to simple basic numbers. Also, Figures 5 and 7 were related to the way of counting horizontally by equal groups of 5 and 10. And the way used in Figures 8 and 9 were related in that the groups were counted by parallel lines to the diagonal. But it was doubtful whether students had followed the way of counting based on the rule given in Figure 8.

The difficult point here was that, in spite of advanced instruction by the teacher, about half of the students counted twice or missed. For the purpose of overcoming this difficulty, the idea of grouping was developed. The next difficulty is that, students use repeated addition, except 5 or 10 in the case of grouping, because they didn't know multiplication beyond the multiplication table. Accordingly, they also made mistakes in counting. Except for the case of adding groups of 5 or 10, they all made mistakes. Moreover, the way of counting in Figure 9, especially, the way of counting to apply the rule for making addition easy after finding it proved to be difficult for this grade. For instance, it was difficult for them to find the pattern of 5x5x2 mentioned above, that is, pair of 5 columns x 5 rows sets.

The Case of 5th grade students of elementary school

Students in the 5th grade were asked to rank 4 major school subjects, National Language, Arithmetic, Social Studies and Science, in the order of their preference before solving problem 2.

Table 1 shows numbers of students by sex who ranked arithmetic as 1st, 2nd, 3rd, and 4th, with the number of boys in the second row, the number of girls in the third row, and the total number of all in the fourth row. From table 1, you could see that more boys preferred arithmetic in comparison with the girls. As a whole, it seemed that there were many students having a favorable impression on arithmetic.

Table 1 Arithmetic Order

order / sex	1st	2nd	3rd	4th
boy	9	4	8	1
girl	4	7	4	3
total	13	11	12	4

Secondly, the computational problem was presented as follows.

PROBLEM 2

(1) Do the following addition by as many different ways as you can. Encircle the best way of all that you think of.

1+2+3+4+5+6+7+8+9+10

Since the pattern of the problem is the sum of natural numbers 1 to 10, generally, the way of counting to make pairs adding 10 would be appropriate; that is, (1+9)+(2+8)+(3+7)+(4+6)+5+10.

In this case, however, in the respect of leaving 5 out, this is not so much desirable as a pattern. Therefore, as the feature of this problem, it is more suitable to make pairs of 11-sums. It also has generality. Eventually, (1+10)+(2+9)+(3+8)+(4+7)+(5+6) or [(1+2+3+4+5+6+7+8+9+10)+(10+9+8+7+6+5+4+3+2+1)] ÷ 2 would be appropriate.

Some examples of students' answers are as follows:

(1) Adding in Left-To-Right order.

(2) Aside from 10, 1+9=10, 2+8=10, 3+7=10, 4+6=10 and then 10+5=15 and 10x4+15=55

(3) 10+9+(8+1)+(7+2)+(5+4)=10+9x5=55

(4) (1+9)+(2+8)+. . .+(4+6)+10+5=10x5+5=55

(5) (1+10)+(2+9)+. . .+(5+6)=11x5=55

(6) (1+2+3+4+5+6+7+8+9+10)+(10+9+8+7+6+5+4+3+2+1)=11x10=110 110÷2=55

Some answers of students were of the same type. For instance, making pairs of 9-, 10-, or 11- sums and then calculating with multiplication. Most of the students, however, found the answer by (1) to start with and then began to divise some others in (2)-(6). They regarded making pairs of 11- sum such as (5) as the best. Next it was good and suitable for them to make pairs of 10- sum. Moreover, there was only one student who used the way of (6), who first used the way of (1) and then secondly used the way of (6).

(2) Add ten odd numbers from 1 to 19 by as many different ways as you can.

The pattern of this problem is somewhat more complicated than the problem described before. In other words, it is the sum of the ten odd numbers 1 to 19. In addition, this problem is represented only in sentences.

The features of this problem are the way of making 10- sum as said before or making 20- sum such as 1+19, 3+17, and so on. Moreover, the pattern of this problem lies in the following.

1+3=4=2x2, 4+5=9=3x3, 9+7=16=4x4, 16+9=25=5x5

25+11=36=6x6, 36+13=49=7x7, 49+15=64=8x8, 64+17=81=9x9, 81+19=100=10x10

Now, we will look over the modes of students' action as follows:

Though the problem 2-(2) is described by sentences without numerical expression as used in problem 2-(1), the intention of the problem was well grasped by all like the problem 2-(1). A couple of students, however, had some mistakes in calculation because of larger numbers. In this respect the problem 2-(2) is different from the problem 2-(1) where all students answered correctly.

Most students applied the same strategy as problem 2-(1) in the solution of problem 2-(2). There were a few students who found the following new pattern.

(1) (1+9)+(3+7)+(5+5)+(7+3)+(9+1)+10x5=10x5+10x5=100

(2) Calculation in Left-To-Right order. Most of the students performed like this:

1+3=4, 4+5=9, 9+7=16, 16+9=25, 25+11=36, 36+13=49

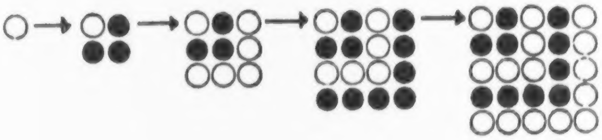
There was, however, only one student who found the following pattern.

1+3=2x2, 4+5=3x3, 9+7=4x4, 16+9=5x5, . . .

81+19=10x10

This idea may be represented as follows.

Figure 11 Square Numbers



The problem 2-(2) was different from the problem 2-(1), that is, some students made mistakes in computation because of larger numbers than those used in the problem 2-(1). But it seemed that all of the students understood the meaning of "odd" number.

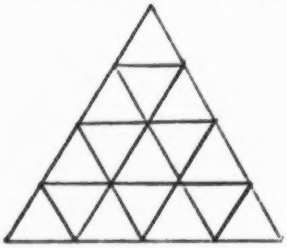
Next, in order to study the difficulties in students' problem solving, problem 3, which seems rather difficult for elementary school students, was assigned to the same subjects.

PROBLEM 3 (Figure 12)

A larger regular triangle is formed by arranging smaller ones, the length of whose side is 1-cm, as shown in Figure 12.

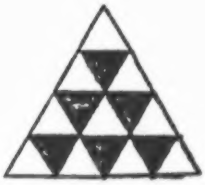
Answer the following questions.

Fig. 12 Regular Triangle



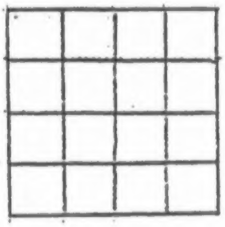
(1) How many regular triangles with 1-cm. sides are there?
(2) How many regular triangles with 2-cm. sides are there?
(3) How many regular triangles with 3-cm. sides are there?
(4) How many regular triangles with 4-cm. sides are there?
(5) Count the number of all regular triangles of different sizes.

Fig. 13 Gestalt of Triangle



The pattern of the problem here will be made clear when compared with the problem below (Figure 14), as to how many squares there are. That is, the relative orientation is definitely only one. But, the feature of this case lies in that the number of two kinds of regular triangles such as △ and ▽ should be paid attention as in Figure 13.

Fig. 14 Square



The number of regular triangles with 1-cm. side is the sum of these:

△ ; 1+2+3+4, ▽ ; 1+2+3

This is the feature of this problem.

More than half of students answered correctly in this problem 3-(1). The strategies for this problem which led to the correct answer are the following:

(a) The case of counting one by one.

(b) The case of 4x4 because of four in each side.

(c) The case of the sum of these; △ : 1+2+3+4 and ▽ : 1+2+3.

In other cases, the answers were wrong.

When the above strategies were applied to the problem 3-(2), those which could lead to the correct answer were (a) and (c) but not (b). The strategy (b) was the one that many students had taken in counting the numbers of square with 1-cm. side in the Problem shown by Figure 14. Students who applied the strategy of (b) answered in the pattern of 3x3, 2x2, 1x1 on and after the problem 3-(2). Further, some students, in the case of (a), failed to count in the process. Especially, at the problem 3-(2) and (3), there were many students who made mistakes.

A few students answered correctly in problem 3-(1) through (5) and came to the strategy applicable for the cases where the side of triangle is divided equally into 3, 4,.., parts; that is, triangles can be grouped by their relative orientations, △ and ▽ , and their size, and numbers of triangles in each group can be counted by addition of numbers in a definite pattern. After all, it was important for students to find the former.

Moreover, the following three facts seem worthwhile to be noted. First, the interesting strategy which had once led to success was likely to go on in a definite direction and could hardly be exchanged midway even when the answer was not correct. Almost all students had retained the images of square areas. For instance, there were many students who answered the above by applying the strategy which led to success in the Square Problem (Figure 14) and problem 3-(1).

(1) 4x4=16 16 ÷ (1x1)=16 The number : 16
(2) 4x4=16 16 ÷ (2x2)= 4 The number : 4
(3) 4x4=16 16 ÷ (3x3)= 1 ... 7 The number : 1
(4) 4x4=16 16 ÷ (4x4)= 1 The number : 1

Second, students were likely to return to the familiar, well-remembered strategy in case they found difficulty. That is, most of them set about the strategy of counting one by one in the troublesome situation. Then, they tried to find a better strategy after getting a solution by that familiar strategy.

Third, only a few students tried to find a new strategy after their initial solution. Most of the students seemed to be satisfied with their first solution even when it was poor.

Implications

The instructional implications from my study on Pattern-Finding in Problem Solving consist of the following proposals:

(1) For the study of students' strategies and difficulties on problem solving, we should consider both the pattern of problem and the mode of the student to do problem solving. We suggest the former has a great effect on the latter.

(2) Some students can solve the problem when they find out the underlying pattern in the problem. The good problem solver easily finds out the pattern of the problem and directly finds the solution.

(3) Otherwise, many students cannot solve the problem in the best way when they do not find out the underlying pattern in the problem. They often go back to their familiar way of solving which they had learned before or they use the trial and error way of problem solving.

This paper has suggested two ways of research that could be used in order to follow students' strategies and difficulties on mathematical problem solving. My hope is that this paper will help to stimulate additional research on these issues.

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Discussion of Professor Nohda's Paper

Clarkson: Now we'll start the discussion. Are there any questions or comments?

Becker: This is a question directed to other Japanese delegates also. Early in your paper, Professor Nohda, you mention you are considering two cases, one of them was the underlying pattern in the problem. The other one is the failure of strategies in students' problem solving. I am wondering whether other Japanese are doing research along this line and making it a focus of study?

Nohda: This is a new point of view. In preparing my paper for this seminar, I took that point by considering my experiences hitherto together with Sowder's opinions expressed in his paper in the reference. The study of understanding or strategy on the part of children for problem solving has not been done hitherto in Japan. So I am now in the process of developing such an approach of study.

Shimada: The viewpoint of Nohda's research is very new in Japan, though there are some papers appearing in the JSME journal reporting the variety of children's activities when they are faced with such kinds of problem solving. The new point is its orientation of observed behavior from this viewpoint.

Becker: I like the problems which Nohda is investigating and the approaches which are revealed about children's thinking. I also want to comment that some of these problems are frequently used by secondary mathematics teachers as good problem exercises in the U.S. And it is interesting to see that simpler cases of them are being looked at with children in Japan.

Kantowski: Professor Nohda looks at individual solutions of student's problem solving. What I would like to know is whether these solutions are collected through paper-and-pencil solutions to the problems or if students are actually interviewed and asked how they solved the problem, and how they thought about problems.

Nohda: I taught a class as a demonstration of teaching and another elementary school teacher, who was a collaborator in my research, taught another class. During these lessons we observed children. Next, we gathered all of the children's papers used in the lesson and studied them. From these, we studied children's thinking processes and strategies. In cases where we were not sure, we interviewed each child to ask how he/she thought about the lesson.

Kantowski: Is this kind of research, of looking at individual children's work, common in Japan or is most of the research done by simply collecting the results children wrote for their answer?

Travis: Had the students been given instruction on pattern finding? Was there a preset classification scheme, or did it come after you studied children's reactions?

Nohda: The collaborating teacher attempted always to teach the heuristic of pattern finding. In response to your second question, it is as I described previously.

Shimada: I will supplement what Nohda has said. At the beginning of his lesson, he gave a paper to each child on which the problem was printed with much space to be used by the children. He asked them to use the paper only in this lesson, and not to write in any other material. After the lesson, these papers were collected as the first set of data. The second approach to collecting data is to observe children while they are working and to take notes about these observations. The third set of data comes from individual interviews with those children whose reactions are not clearly understood by the observer (on how they are thinking during the lesson, or how they thought about the lesson, after the lesson). Collected papers are useful because children are accustomed to express their ideas of how to group objects by writing expressions or by drawings on the figure in the given problem, even when they count the numbers one by one.

Nohda: In the paper-and-pencil results in the classroom, the students are trained to write equations or indicate some way of how they solved the problem or how they grouped the objects when they were required to count the number.

Clarkson: Professor Travis, does that answer your original question?

Travis: I was looking at the different figures used to classify the way that students solved problems. Did you already have those figures in mind or were these just ones that students developed themselves in their problem solving process? So was it a preset classification scheme with your 9 or 10 figures or was it something the students generated through their processes?

Nohda: Yes, the students make these figures themselves.

Ishida: In implication Dr. Nohda splits the thing into the pattern of the problem and the mode of the student in the problem solving. I want to ask you to give specific examples.

Nohda: Look at Figure 1 and Figure 1C together. That's the general characteristic of this particular problem. The most widely used method among students is this counting one by one method. The second most common method used among students is the counting in pairs. The next most common are counting by 5's and by 10's. Counting, as you see in Figures 5 and 7, is usually carried among very high level students. Only two or three students did it as you see in Figures 8 and 9. In discussing these with the students themselves, the consensus reached among them was the methods seen in Figures 5 and 7. Thus, when teaching this in the classroom I stressed these two methods.

Wilson: In the triangle problem, in teaching the triangle problem, you called their attention to counting triangles right side up and triangles upside down. Those are separate countings. Have you taught it in a manner in which you let them come to or discover that? Have you tried the alternative, that's the key to generalizing the problem? Have you tried letting the students find that pattern?

Nohda: I don't teach them some of the count upside down ones and so on. The students make the distinction by themselves. They do the classifying.

Kantowski: Professor Nohda, you mentioned earlier that you teach the class or one of your assistants teaches the class. Is this a class that you work with often or was this a class that you worked with for the purpose of this experiment? Are these students that you often visit and you often teach, or is it just for this experiment that you worked with them?

Nohda: I do not usually have contact with these students, because they are not students of my usual class. The students are taught by a teacher who often comes to me to discuss problems in teaching. Then in order to demonstrate teaching of problem solving, I go to the teacher's class and that is the sort of situation that I work in. So it is just some schools that I go to.

Silver: I would like to ask Professor Nohda what he sees as the goal of this research, why does he wish to conduct the research in this way, studying in classrooms with students, but studying individual processes and so on.

Nohda: The overall purpose or the goal of my research is to study the problem solving process and to see how students understand problem solving, and then to follow the steps by which they come to this understanding. Again, the purpose is to follow through the internal workings of the students and how to observe or how to be able to pick this out from without, externally.

Silver: As a follow-up comment, I would just like to say that I think I would like to encourage this kind of inquiry to continue in Japan. I am delighted to see that it's happening. It is very consistent with many of the kinds of research that are being done in the U.S. and is, in fact, very similar to research that Dr. Kantowski and Dr. Rachlin have done in their own work with students. I think this provides a kind of experimental basis for us to think about some studies that might be done cross-culturally.

Clarkson: Thank you very much, Professor Nohda. We enjoyed your talk and thank you for the discussion.

Professor Becker's Paper

Sawada: I would like to introduce Professor Becker. He is a Professor of Mathematics Education in the Department of Curriculum and Instruction at Southern Illinois University at Carbondale. He is co-organizer of this Seminar. He is well known in mathematics education community not only in the U.S. but in the world. He is also well known to scholars in the field in Japan, as he has visited Japan several times, and made lectures in the first U.S.-Japan Seminar on Mathematics Education in 1971 and in the Regional Conference on Mathematical Education in 1983, both held at Tokyo. He has done many projects and published many books. Recently his major interest is problem solving, especially non-routine type.

WHY ARE STUDENTS UNABLE TO SOLVE NON-ROUTINE PROBLEMS IN MATHEMATICS?

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Introduction

A great deal has been written about problem solving in mathematics in the United States in the last few years. Position papers and authoritative reports have been issued by the National Council of Supervisors of Mathematics (1977), the National Council of Teachers of Mathematics (1980, 1981) and the Mathematical Association of America (Schoenfeld, 1983). NCTM's An Agenda for Action (1980) suggests that:

- The mathematics curriculum should be organized around problem solving.
- The definition and language of problem solving in mathematics should be developed and expanded to include a broad range of strategies.
- Mathematics teachers should create classroom environments in which problem solving can flourish.
- Appropriate curricular materials to teach problem solving should be developed for all grade levels.
- Researchers should give priority in the 1980's to investigations into the nature of problem solving and to effective ways to develop problem solvers. (pp. 2-5)

The positions set forth in the various reports are reinforced by the results of the National Assessments of Educational Progress (NAEP) (1980, 1983) and the Second International Mathematics Study (SIMS) (McKnight, et al., 1986). All show evidence that mathematics achievement is low in the U.S. The NAEPs and SIMS indicate that performance on non-routine problem solving of 9-, 13-, and 17-year olds to be unacceptably low. The

National Commission on Excellence in Education report (1983) calls for increasing standards, more mathematics for students and increasing testing and monitoring of student achievement.

These achievement data and recommendations come as no surprise to many teachers for they have observed for many years that problem solving has been difficult for students to learn and teachers to teach. Teachers frequently hear students at all levels ask "Do I add or subtract?" or "Do I multiply or divide?" and show little thinking about the problem at hand. They also see students get "solutions" to problems which seemingly bear no relation to the problem being solved. The development of critical thinking skills is a matter which needs the attention of teachers and curriculum developers.

The Sources of the Problem

Various researchers have been looking at the situation. As one example, Sowder (1986) studied students' strategies in solving story problems in grades 6 and 7. The strategies he found were the following:

Desperation Strategies

Strategy 1: Find the numbers and add.

Strategy 2: Guess at the operation to be used.

Computation-Driven Strategies

Strategy 3: Look at the numbers; they will "tell" you what operation to use.

Strategy 4: Try all +, -, x, ÷ and choose the answer that is most reasonable.

Strategies Using Some Degree of Meaning

Strategy 5: Look for isolated "key" words to tell what operation to use.

Strategy 6: Decide whether the answer should be larger or smaller than the given numbers. If larger, try both + and x, and choose the more reasonbable answer. If smaller, try both - and ÷, and choose the more reasonable.

Concept Driven Strategy

Strategy 7: Choose the operation whose meaning fits the story. (pp. 3-4)

Sowder found that strategy 7 was rarely used, even by the better students. Missing in the curriculum seems to be students' ability to think about what they are doing. These results are based on one-step problems and thus, the implications for students' performance on multi-step or non-routine problems seems very clear.

Whitney (1985) writes about the same phenomena after doing an analysis of student performance on NAEP test items. He concludes that in these kinds of school problems (not real life problems) students just guess at an operation and rarely look or check for meaning. The data show strikingly how schooling does not serve to help children see reality (p. 221). Moreover, he sees that the same is true even on "skills" problems that are emphasized throughout the curriculum. According to Whitney, the problem, a fundamental one, is the way in which schooling is organized and the manner in which students are taught.

Lockhead (1980) writes along these same lines. He says that poor problem solvers (and there are many) are not "active" because they do not see that there is anything for them to do:

> Their view of both problem solving and learning places them in the passive role of absorbing information and giving it back. They think you either know the answer to a question or you don't. While this attitude may seem naive, it is in fact the logical consequence of most schooling. (quoted in Stonecipher, 1986, p. 41)

One important component in problem solving, of course, is translation from problem statements to equations. Clement, Lockhead and Monk (1979), Rosnick (n.d.), Wollman (1983) and others have done extensive research on this and have found the reversed order of variables in equations to be a significant problem. For example, it is common for even good students at the university level to write $6S = P$ as the equation for:

> At a certain university, for every professor there are six students. Write an equation using P for the number of professors and S for the number of students.

Similarly, students too frequently write $4C = 5S$ as the wrong equation for:

> At Mindy's restaurant, for every four people who ordered cheesecake, there are five who ordered strudel. Let C represent the number of cheesecakes and S represent the number of strudels.

Researchers have found this reversibility problem to be deeply entrenched in students problem solving behavior. But why should such a problem exist for students? One possible source of the difficulty is the manner in which students use key-word matching (or syntactic translation) in translation. Students learn in school to map key words of the problem statement directly onto mathematical symbols as follows:

> A number diminished by 3 is equal to 16. The mapping follows:
> a number diminished by 3 is equal to 16
> $x \quad - \quad 3 \quad = \quad 16$

In this problem the method works, but as a general method it is useless. The reader can readily see where $6S = P$ comes from in the student-professors problem.

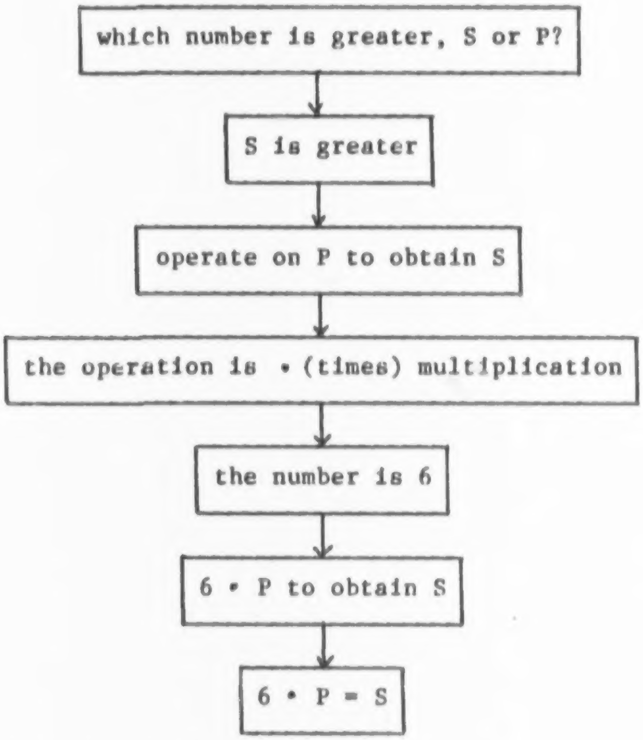
Rosnick (n.d.) did a textbook review of junior high mathematics texts and found that 11 of 12 textbooks used the key-word matching approach. Similar results were found in 7 of 11 first-year algebra textbooks and all 5 second-year algebra textbooks. It is clear that, in general, school textbooks use this method and then present a multitude of problems for which the method works. This âpproach, used at all grade levels and with little attention given to thinking about the problem, certainly must contribute towards jeopardizing students' potential as successful problem solvers.

Another method, termed Figurative Translation, aids the student in seeing from a reading of the student-professor problem that the student population is larger than the professor one. Perhaps the student even makes a diagram such as:

> six students for every one Professor
> SSSSSS P

But the student still writes 6S = P rather than the correct equation 6P = S.

A third method characteristic of students who can solve problems correctly is Operative Translation. In the student-professor problem, the student thinks about the situation and sees that S is greater than the number P and thus it is necessary to operate on P in order to produce S as is shown in the following flow chart (Schrader, 1985, p. 13):


which number is greater, S or P?
S is greater
operate on P to obtain S
the operation is • (times) multiplication
the number is 6
6 • P to obtain S
6 • P = S

Rosnick (1980) found data to support the hypothesis that students tend to view letters in equations as labels for concrete entities rather than the number of things or objects. Further interview research by Clément (1982) and a study by Schrader (1985) lead to the same conclusion. Students have acquired profound misconceptions that are difficult to overcome.

Rosnick (1980), Wagner (1983), Wollman (1983) and Schrader (1985) give important suggestions concerning how we can help students to develop a better understanding of symbols and work has been done (e.g., Wollman & Schrader) to show how the reversibility problem can be overcome by using pair-problem solving. This method involves two students working together in solving problems, interchanging roles in alternating fashion through a set of problems: One student solves the problem thinking aloud and the other listens and constantly checks for accuracy and demands constant verbalization of the other's thinking. Students are not permitted to be passive and the teacher observes and provides feedback -- slow, careful work is emphasized with frequent checking of results. There are other important components of this teaching strategy but the main point here is that students have misconceptions about problem solving that come from, to use Whitney's terminology, school learning and not the reality of the student. From the textbook and students' perceptions of what the teacher expects, misconception flows.

It seems clear that research is needed aimed at a better understanding of children's developing capabilities for scientific reasoning. Carey (1985) has found that young children are capable of using quite intricate thought processes and, for this reason, Mathematics, Science and Technology Education (1985) recommends this as an area for further research. This recent book from the U.S. National Research Council, in fact, lays out recommendations to the National Institute of Education for extending and utilizing research to improve school education. Carey's finding is supported by other noted writers such as Papert (1980) and Whitney (1985).

Our efforts to deal with the problem at the secondary level may be doomed for the attitudes and approaches of students are already deeply entrenched. The origin of the problem occurs at the early elementary school level and it is at this level that intervention has to occur. If not, the die has already been cast by the time students reach the junior high school level.

In early childhood "children explore their environment and learn in manifold ways, at a rate that will never be equalled in later life; and this with no formal teaching" (Whitney, 1985, p. 222). The broad manner in which young children learn no doubt occurs because they have freedom to think with wide latitude and with a natural curiosity and flexibility. At this age level children gather and process information in ways many teachers are unaware. They think about and consider numbers in a natural way (Ginsburg, 1977). But it seems that we do not facilitate and nurture this mental behavior once the children reach school. Whitney comments

> How can it be that when preschool children think so naturally and successfully, in school they get pulled into dropping such thoughts and trying to think only as they are told?...the children's natural thinking, with looking for meanings, becomes gradually replaced by attempts at rote learning, with disaster as a result. And the more the pressures are applied to enforce learning, the more its rote character is fixed, resulting in further failure. (p. 233)

Easley and Easley (1982) believe that students' difficulty emerges very early when they first encounter counting and the addition algorithm. Whitney contends that the difficulty begins when they encounter subtraction of two-digit numbers when, because of the emphasis placed on the algorithm, students begin to drop meanings and look only at the patterns of digits as in:

$$\begin{array}{r} 42 \\ -\ \underline{18} \end{array} \quad \text{to} \quad \begin{array}{rr} 4 & \quad 2 \\ -\ \underline{1} & \underline{\quad 8} \end{array}$$

It is natural to take 2 from 8 but now students have to "borrow" from the 4 in order to do the subtraction (not a natural thing). Meaning goes on the wane and rules learning and complexity set in along with conflict which leads towards anxiety. Rote learning ensues and is worsened by the experience of dealing with fractions later and, still later, long division. This attitude, void of understanding the meaning of what they are doing, is further worsened by the emphasis on skills development. By the end of the elementary years, Whitney believes children have a firmly entrenched attitude that "school math is something for itself, not for life outside school" (p. 226). A manifestation of this attitude is well known to teachers who hear students ask for "the formula you want me to use" and, in particular, rebel when the teacher encourages students to "think about what you are doing." The attitude is further worsened when, as we have seen, students are wrongly taught in learning algebra to translate using the Key-Word Matching approach. It, too, is devoid of meaning but, yet, students seemingly "succeed" in doing a multitude of problems in which that method "works." Thinking clearly falls by the wayside and the result is that problem solving suffers and attempts to improve students' problem solving abilities will at best be extremely difficult in the secondary years.

This state of affairs was addressed by Benezet, Superintendent of Schools in Manchester, New Hampshire, in the late 1920's (1935a; 1935b; 1936). In his schools he observed many of the difficulties discussed earlier in this paper. He observed that "For some years I had noted that the effect of the early introduction of arithmetic had been to dull and almost chloroform the child's reasoning faculties" (1935b, p. 242). After devising a plan to deal with citizens' deeply rooted prejudices which favored arithmetic beginning in grade 1, he devised a new course of study which delayed the teaching of formal arithmetic for several years including, instead, a great deal of thinking exercises embedded in the reality of children. Instruction was designed to avoid developing an attitude that a formula or fixed method can be used as a substitute for thinking. The basic computational algorithms were not taught until the sixth year and when started, emphasis was placed on understanding and avoidance of purely mechanical drill. In no case did speed or "covering ground" take precedence and emphasis was placed on mental as opposed to written activity. Before starting on any computational problem, students were required to estimate the answer and the final result was always compared to the preliminary guess. Teachers were trained to guard against letting their teaching degenerate into mechanical manipulation without thought.

The differences between Benezet's traditionally taught children and the experimental groups were striking and showed clearly on fifth grade children's solutions to the following problem which was represented in a diagram.

> Here is a wooden pole that is stuck in the mud at the bottom of a pond. There is some water above the mud and part of the pole sticks up into the air. One half of the pole is in the mud; 2/3 of the rest is in the water and one foot is sticking out into the air. Now, how long is the pole? (1936, pp. 7-8)

Traditional groups. First child: You multiply 1/2 x 2/3 and then you add one foot to that; second child: Add 1 foot and 2/3 and 1/2; third child: Add the 2/3 and 1/2 first and add the one foot; fourth: Add all of them and see how long the pole is; next child: One foot equals 1/3. Two thirds divided into 6 equals 3 times 2 equals 6. Six and 4 equals 10. Ten and 3 equals 13 feet. Similarly for other students. None saw the essential point, namely that 1/2 the pole was in the mud and the other half was above the mud and that 1/3 of this half equaled one foot. The only thought was to take the numbers and manipulate them. Next child: One foot equals 3/3. Two thirds and 1/2 multiplied by 6. Question: Why do you multiply by 6? Answer: OK, divide. Question: How much of the pole is above the mud? Expected answer: One half of it is above the mud. Given answer: One foot and 2/3. And so on...

Experimental groups. Given the same problem and diagram, first child: You would have to find out how many feet are in the mud. Question to another child: And what else? How many feet are in the water and add them together. Question: How would you go to work and get that? There are 3 feet in a yard. One yard is in the mud. One yard equals 36 inches. If 2/3 of the rest is in the water and one foot in the air (one foot equals 12 inches), the part in the water is twice the part in the air so that it must be 2 feet or 24 inches. If there are 3 feet above the mud and 3 feet in the mud it means that the pole is 6 feet or 72 inches long. Seventy-two inches equals 2 yards. The child translated into inches because she could not believe that she was doing all that was necessary by simply saying the pole was 6 feet long. Giving 72 inches as an answer made it hard enough to justify asking the question! Next child: One half of the pole is in the mud and 1/2 must be above the mud. If 2/3 is in the water, then 2/3 and one foot equals 3 feet, plus 3 feet in the mud equals 6 feet. The problem seemed simple to these children who had been taught to think about the situation.

These results were borne out in a number of other problems requiring reasoning. Elementary school children in the experimental group easily outperformed ninth grade students in the traditional curriculum. It was also noted that children in the experimental group developed a readiness and fluency in language that was surprising and children, in general, were better able to express ideas in words.

What Can Be Done?

Much of the problem we face in improving problem solving abilities rests with changing students´ attitudes towards learning and mathematics. The problem should be addressed during the early formative yars in which emphasis is placed on meaning and reasoning. Benezet, Whitney, Easley and Easley (1982) and others have concrete ideas on which we can follow up.

Freudenthal (1985) also offers some insights relevant to elementary schooling in mathematics. We are all aware that number is an important mathematical tool. Indeed school mathematics is concerned with the development of the real number system beginning with natural numbers at the beginning of schooling. But Freudenthal raises the question whether ratio and proportionality actually precede the development of number in children.

Children recognize pictures of animals, boats, automobiles, etc., as images of physical objects, regardless of the scale of drawing used. Further, they regularly accept a picture the teacher sketches on the blackboard as ten times larger than that on an assignment sheet. They similarly accept units of the number line on a floor compared to one on the assignment sheet. However, when structural changes are made that violate similarity between the two, children will likely raise a question. In Freudenthal´s words, "what is mutually equal in the original, should be mutually equal in the image, which implies invariance of internal ratios, characterizing mappings as similarities" (p. 4).

Why then has the "number string" overtaken the "ratio and proportionality string" in the elementary school? The answer has to do with verbalization which occurs more easily with the number string. Ratio and proportionality is taught in the late elementary grades when we think children can more easily verbalize it. However, this is very late in a child´s education if, indeed, a child has an internal ability to handle the concepts intuitively much earlier. We might wonder whether this helps to explain why so many people never really learn to understand ratio and proportionality at all.

Understanding ratio and proportionality can be facilitated by visual and numerical representations and, thus, be verbalized more formally in the language of geometry and arithmetic. It would be better, perhaps, to connect or integrate the two strings with each other and link them with the same aspects of a child's reality and extend them as that reality changes. Streefland at the IOWO curriculum development project in the Netherlands has been working on this. He emphasizes the intertwining of the number and ratio and proportionality strings by developing a teaching unit for use at the third grade level. This research seems quite important. Space does not permit an extensive description of this work, but the following example may help to illustrate what the IOWO researchers have in mind. Six lessons are described by Freudenthal under the title "The Greetings of a Giant":

> A window of the classroom was open. The blackboard shows the traces of a big hand. A giant must have been in the classroom. Giants are tall. But how tall was this giant? "Look at my hand." The teacher puts her hand on the trace of that of the giant, which appears to be four times as big. The teacher is being measured. A string four times as long as the teacher's height is cut off. The children write a letter to the giant on the blackboard. "This is your height." Next day the giant has answered the letter. It is difficult living for a giant in a world of people Why is it difficult? Where can a giant go and where not? How many sandwiches will he eat? The teacher's shoe is being measured on a piece of paper. There are various reactions as to how big the giant's foot is. Finally they try to fit in 16 teacher's feet to get one of the giant.
>
> The baker found strange footprints in his garden. He called the reporter of the local newspaper, who took a picture. The children discover human footprints, the giant's footprints and intermediate ones, which are interpreted as those of the giant's son. The reporter wrote about it in the newspaper.
>
> The giant noticed it. But true giant's newspapers are larger. How large should they be?
>
> Again footprints, now in the snow. The children compare them by proportionality tables. Baking a cake for the giant. What size should it be? It is cold. A mitten for the giant. What size should it be? How long would it take to knit it? How many balls of wool? An opportunity for a lot of proportionality tables.
>
> Well, you might ask me: Is this reality - a giant's world? Yes, it is. It is as much reality as any good fiction is. And this is, believe me, good fiction for 3rd graders, as Gulliver in Lilliput, which we adapted to be used in the 5th grade. On the contrary, traditional word and application problems are bad, very bad fiction. (p. 8)

This is but one glimpse at an extended curriculum which attempts to intertwine learning strings. In the process, numerical problems are kept to a minimum and emphasis is placed on visual geometry to which students can relate, and thinking and reflecting in the reality of students' thoughts are emphasized. As an aside, it might be mentioned that IOWO researchers deal with students at the seventh year level (coming from many different schools) who differ enormously in their skill and understanding of arithmetic which most of them hate and believe they are incapable of learning.

Sample Problems Which Illustrate Thinking and Operative Behaviors in Problem Solving

The examples given earlier illustrate the kinds of thinking problems with which children need experience in order to develop reasoning abilities. My idea is that we need to provide much such experience for students consistent with the notion that:

> Mathematics teaching is doing mathematics with students which fosters the operative behavior of looking for patterns, discovering, getting insight, developing flexibility of thought and making generalizations.

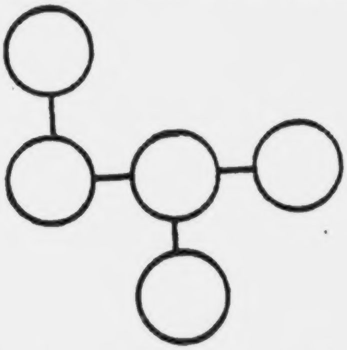
The examples below provide much such experience and generate a lively problem solving context. They are but a few of many such problems I have used both in school classrooms and with pre-service elementary teachers in the university.

Example 1 (Early Primary) Friends (Easley, 1982)

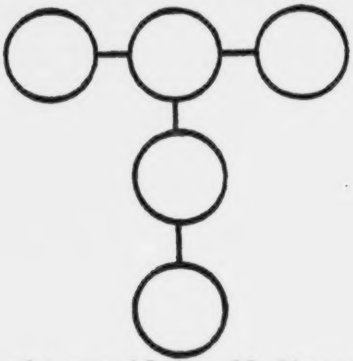
Some friends come to play with me. First 1 came. Later, 3 came. How many of us were there altogether?

Example 2 (Primary School) Number Puzzles (May, 1985)

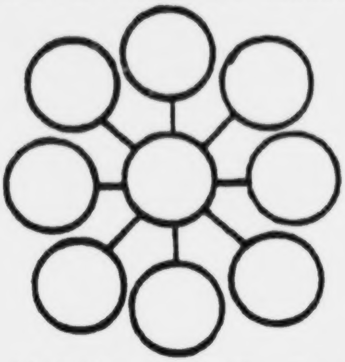
Given chips with the numerals 1-9 on them and a puzzle board: Use chips 1, 2, 3, 4, 5 to make a sum of 6 in all directions.



Use chips 2, 3, 4, 5, 6 to make a sum of 13 in all directions.

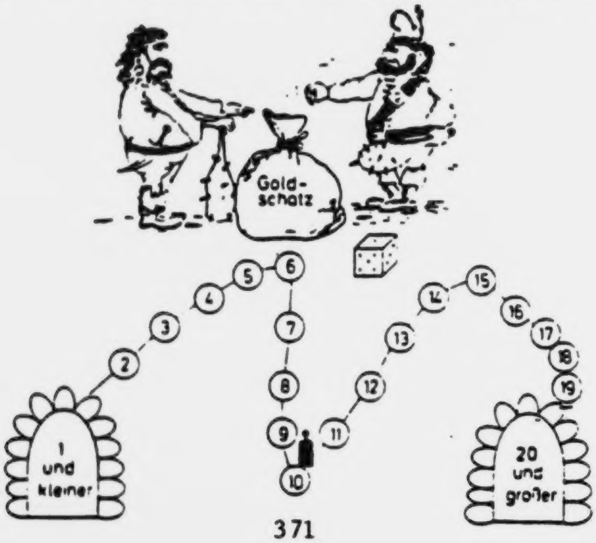


Use all chips to make the sum 15 in all directions.



Example 3 (Primary Level) Robbers and The Treasure (Whitman, 1985)

Two robbers are fighting for a treasure. After some time there is no winner and they are exhausted. So they agree to decide the quarrel by playing a game: They paint fields 1 to 20 between their caves. The treasure is put on field 10. Now they throw a die alternately. According to the result the treasure is moved towards the corresponding cave. As soon as the treasure enters a cave, the owner of the cave wins it.


Gold-schatz
1 und kleiner
20 und größer
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19

Play the game with your partner.

Now suppose the treasure is on field 11: Where might it be after each of the two robbers has thrown the die just once? Where will the treasure be if the right robber throws a "5" and the left robber a "4."

Example 4 (Primary Level) Ice Cream

An ice cream seller offers four kinds of ice cream: Vanilla, chocolate, strawberry, and pistachio. She sells cones with three scoops. How many different cones are possible?

Example 5 (Intermediate/Middle School) Apples and Oranges (Miwa, 1985).

We wish to buy apples and oranges such that the total number is ten and the total cost is 650 yen. Apples cost 75 yen each and oranges 50 yen each. How many apples and oranges can we buy?

Example 6 (Intermediate/Middle School) Rabbits and Hutches (Burton, 1979).

There are some rabbits and some rabbit hutches. If one rabbit is put in each hutch, one rabbit is left over. If two rabbits are put in each hutch, one hutch is left empty. How many rabbits are there? How many hutches are there?

Example 6 (Intermediate/Middle School) Piles of Pennies

In how many ways can 15 pennies be placed in four separate piles so that no two piles have the same amount?

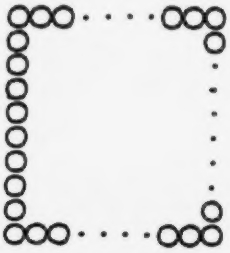
Example 7 (Primary/Middle School) Cats and Birds

Gina and Tom had cats and birds. They counted all the heads and got 10. They counted all the feet and got 34. How many birds and cats are there?

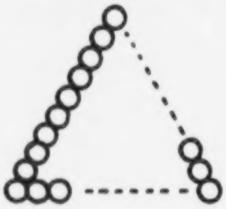
No algebra is necessary and, once solved, larger parameters can be used whose solutions involve a general strategy.

Example 8 (Intermediate/Middle School) Marbles (Miwa, 1985)

How many marbles are needed to construct the square which has 10 marbles on a side?



With 5th through 8th grade students, several different strategies were found for solving the problem. Similarly with preservice elementary teachers. School students frequently concluded the solution is 4 x 10 = 40 and stuck to this conviction until they were shown the following figure at which time they discovered they counted the corners twice in the problem above.



Among 35 5th-8th grade students studied by clinical interview, no student used the general method 4(n-1) for the square nor 3(n-1) for the triangle.

Example 9 (Middle School/High School) Squares (Whitman, 1976)

Look at the following square containing whole numbers

4	7	11	
3	5	8	
7	12	9	
			47

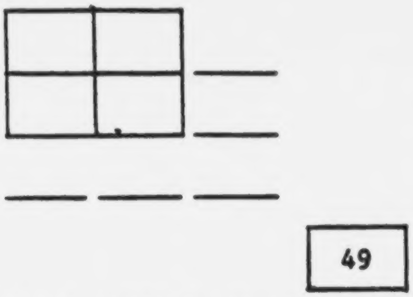
1. Discover the rule for finding the numbers 11, 8, 9, 12, 7.
2. Discover the rule for finding the "boxed" number 47 (called the sum of the square).

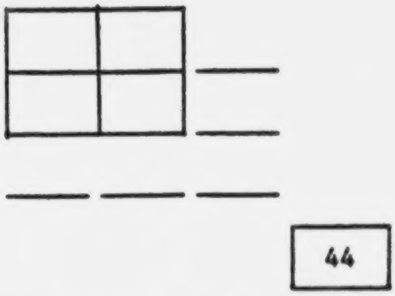
Find the sum of the given squares below:

7	13		
25	8		

8	8		
8	8		

Now change the question and ask students to determine a square with a given sum.

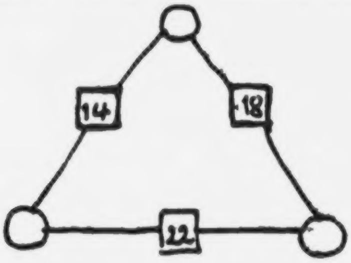




Here the student can find several different strategies for finding a square with a given sum. The student "operates" on the upper right and lower left numbers and on the upper left and lower right numbers to discover the "behavior" of the square. A general strategy can be found and there are several interesting extensions of the problem.

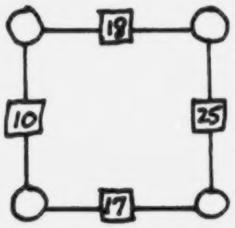
Example 10 (Middle School/High School) Arithmogons (McIntosh & Quadling, 1975)

Given a three-sided arithmogon, put three numbers in the squares -- the number in each square must equal the sum of the numbers in the two circles on either side. Find the numbers for the circle at each vertex.



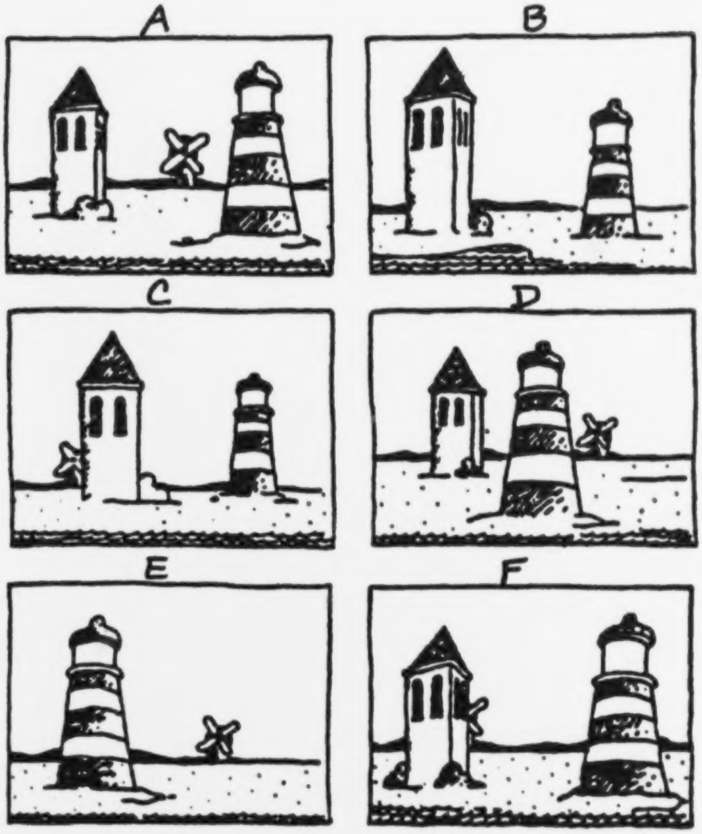
Begin, for example, by placing a "7" in the top circle. Then you must put "7" in the lower left (7 + 7 = 14) and "15" in the lower right (7 + 15 = 22). However, 7 + 15 $\neq$ 18. So change the first "7" to another number and repeat. Look for patterns and a general strategy.

Change to a square arithmogon and try to find a solution(s).



Search for patterns and strategies.

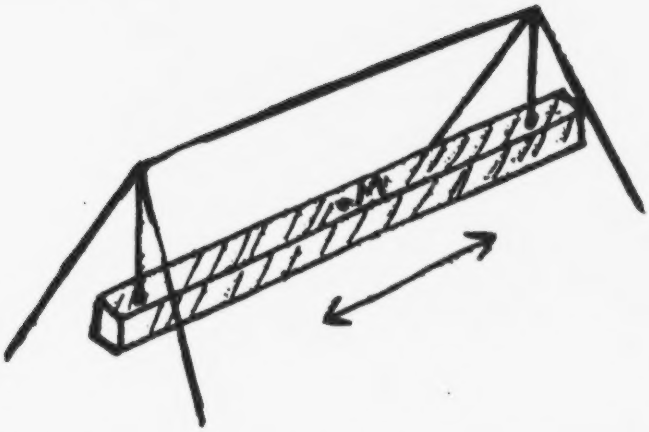
Example 11 (Intermediate/Middle School) Ship (Source unknown)



A ship passes a town - the captain takes a series of photos. Afterwards he accidentally drops them. Put the pictures in the correct order.

Example 12 (Intermediate/Middle School/High School) Log

On children's playgrounds in China one sometimes finds the apparatus pictured below. It is a large log suspended at two points by chains and the log swings to and fro as indicated by the arrow.



A child gets on the log and stands. Another child pushes the end of the log so it swings to and fro. The child on the log tries to walk from one end to the other.

a) When should the child walk and when should she stand still in trying to walk from one end to the other while the log is in motion?

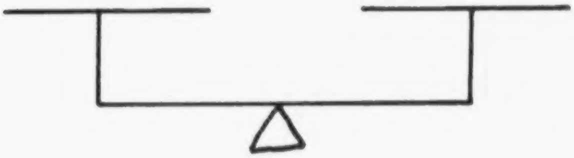
b) Describe the position the log is always in and explain why.

c) Describe the path of point M in the middle of the log and explain.

Note: It may be useful to "act out" what is going on to answer a). A meter stick suspended by two strings which the teacher holds may be useful in answering b) and c).

Example 13 (Middle/Senior High) Balls

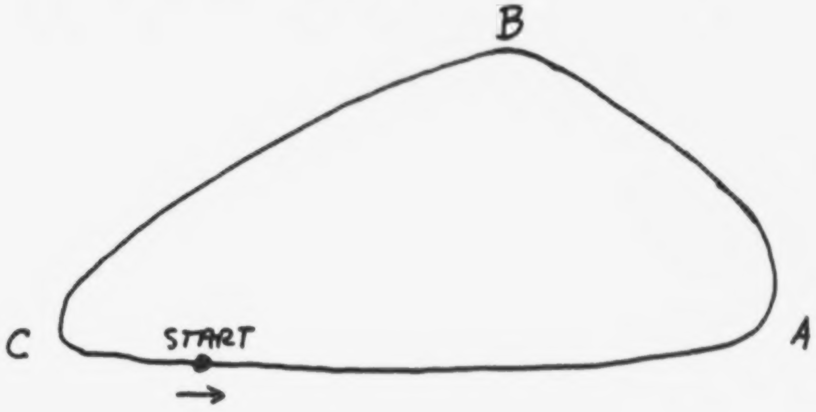
Eight balls are on a table -- all look exactly alike but one is just slightly heavier than each of the others which have the same weight. Given a balance and making no more than two "weighings," can you determine which is the heavier one?

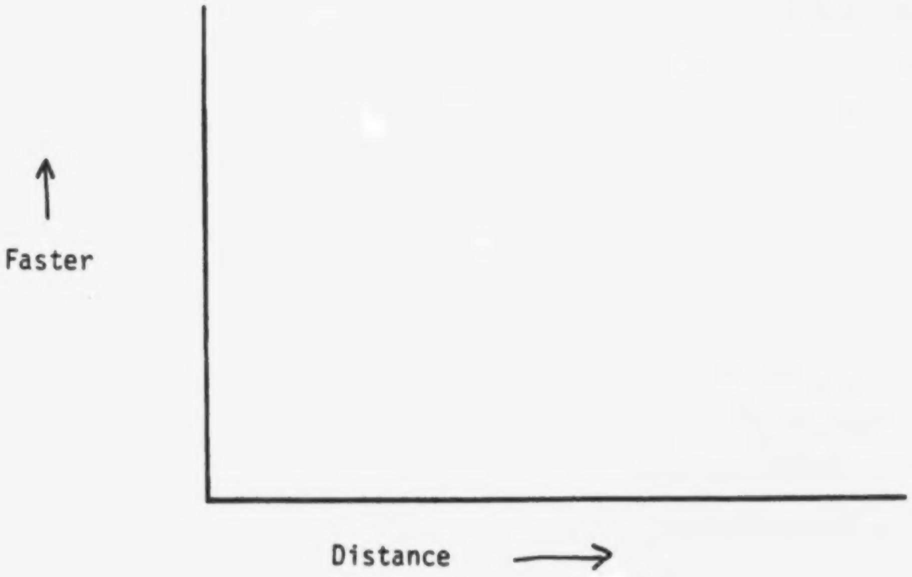


The problem can be changed. For example, suppose we state that the one ball has a different weight, but do not specify lighter or heavier; now, what is the minimum number of "weighings" necessary to isolate the different one? We could also change the number of balls and "weighings."

Example 14 (Middle/High School) Modelling Relationships (Shell Center for Research in Mathematics Education, Nottingham, England)

Situation: Look at the race track illustrated below and imagine a race car on the first lap of a race. Draw a distance-speed graph of the race car going around the track.





The solution will show how considerable growth in understanding of the relationship between a graph and the situation can be developed by sketching a graph and then refining it, rather than attempting to set it "all correct" right away (shows tolerance for lack of closure, an important aspect of problem solving). Students' attention may have to be directed to the following:

number of bends in the track

how the bends are modelled by minima

relative difficulty of the bends

behavior of the car's speed along "straights"

relative lengths of the straights

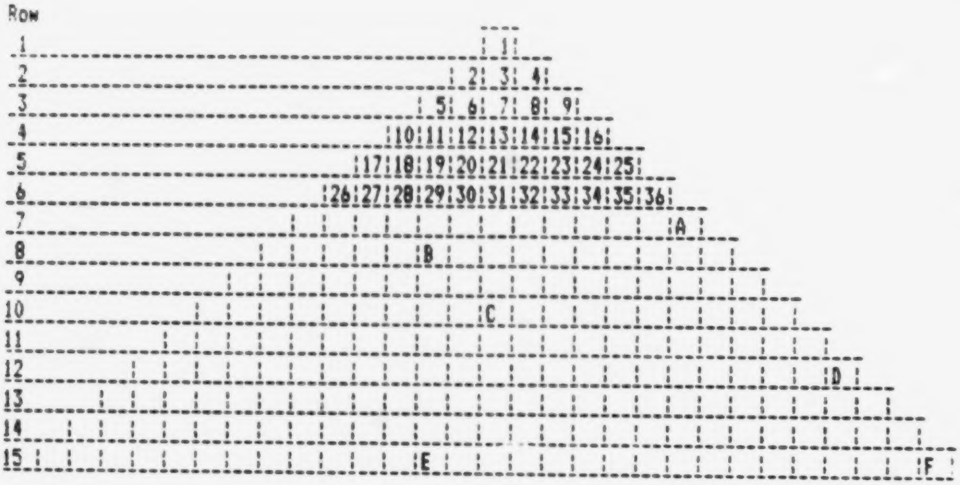
Considering one of these items at a time may help the student to monitor his/her growth in understanding the situation. The shape of the track can be varied to see how the modelling relationship needs to be changed.

How does . . .

1. the temperature of the water in a teakettle vary when the fire is lit under it?
2. your enjoyment of a cup of coffee vary with its temperature?
3. the water level in your bathtub vary before, during, and after a bath?

etc.

Example 15 (Middle/High School) The Number Pyramid (Becker & Beaty, 1986)



Find the numbers you think belong in the blank spaces below. Try to look for patterns in the table that will help in finding the answers.

A = _____ B = _____ C = _____ D = _____ E = _____ F = _____

What do you think will be the last number in row 23? _____

What do you think will be the first number in row 58? _____

List as many patterns and relationships as you can.

Example 16 (High School) Dividing Space G. Polya

What is the maximum number of parts into which space can be divided by 5 planes?

Solving the problem involves spatial visualization, use of heuristics (think of a simpler, similar problem, and looking for patterns). The solution is a surprise to students.

Conclusion

My experience indicates that when more challenging non-routine problems are given to students or to pre-service elementary teachers, interest and involvement is generated - far more so than when "typical word problems" are used. Moreover, I have found that challenging problems bring students together in thinking about the situation, searching to understand the problem, and then trying to solve it. Sometimes I almost sense that a "spirit of community" ensues with students reflecting and building on each other's ideas - this is a healthy state of affairs. I also believe that such experiences help students to develop a tolerance for lack of closure which is fundamentally important in problem solving. This experience may help to break students' natural dependence on key words as a means to jump from the problem statement to a "solution."

Finally, I think it would be good for us to study Easley and Easley's four changes needed in primary school mathematics (1982, pp. 117-124). Their work with primary school teachers is based on extensive observation (15 weeks) in a primary school classroom in Kitamaeno School, a public school in Tokyo. They suggest:

1. Work on more challenging problems.

2. Base all calculations on partitioning and regrouping rather than counting.
3. Organize the class into long-term groups for building confidence.
4. Have children keep notebooks of their own work on challenging problems by writing equations, complete answers, and explanations in their own words.

Each of these is elaborated and they discuss how they are implementing their ideas with American teachers. More elaboration would be useful here, but space does not permit it. Suffice to say, that a research agenda needs to be set to study both problem solving behavior and curriculum development at the elementary level as well as both the inservice and preservice education of teachers.

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Discussion of Professor Becker's Paper

Sawada: Thank you very much for your presentation. Now we begin discussion. Have you any points or questions?

Silver: In your paper, Jerry, you seem to make several different points about what the real problem is with students' inability to solve non-routine tasks. At one point, the argument seems to be that there is some difficulty between the artificiality of school vs. the real world on reality-based tasks. At another point it seems that the argument is that a big part of the problem is emphasis on number rather than ratio and proportionality as the basis for the design of elementary school instruction. But it seems to me that what runs through all of what you've written in the paper and in all of your remarks here, and I would want your comment on this, is that the problem is really that we don't emphasize thinking in mathematics. That, in a sense, that when we present students with these tasks they are qualitatively different in nature from the tasks that students are typically presented with on a day-to-day basis in mathematics class, and that students become accustomed to not thinking when doing mathematics tasks. These are relatively automatic response situations and not ones in which it's valuable to be thoughtful. And would you agree that our curriculum just doesn't emphasize thinking in mathematics as the basis for its design?

Becker: I agree completely that we don't emphasize thinking enough. I appreciate your reaction because I thought that, while perhaps that theme may not have come through in what I've said here, I had intended it to come through in the paper, much of which I didn't read to you. The whole point of the Benezet studies is that kids don't think about what they've been doing. And that set him on a course of developing an experimental curriculum in which he put off the teaching of arithmetic, as we typically think of it, until either the sixth or seventh year of school.

Going back to the reference to number/ratio-proportionality, Freudenthal gave a talk about this a little more than a year ago at a conference at the University of Chicago. His idea strikes me as an intriguing one, and I think that the work being done at IOWO to test out these ideas is therefore very important. It will be interesting to see where it leads.

Silver: Can I just follow up on that? Part of my motivation for asking that question was that I was struck by the fact that I thought that perhaps Whitney might not like Benezet's problem and that Freudenthal might not like Benezet's problem either and that Benezet might not like what Whitney and Freudenthal had to say. But that what they all had in common was an interest in provoking students to think, to think deeply about what they were learning even though they might disagree about the artificiality of the Benezet problem, the fact that how would anyone ever know that exactly half of the stick was in the mud and exactly one foot was out of the water without knowing the length of the stick. And the artificiality of the situation is profound. And yet, it is an interesting problem to provoke thinking and so what seems to be a strand that runs across and does, in fact, unify this is the interest in provoking students to think more deeply and more profoundly about the mathematics they are learning.

Becker: Yes, all three have a common interest in teaching kids to think and to use their reasoning powers. I'm not bothered by the "artificiality" of the Benezet problem, however. Because embedded in it is a critical kind of thinking activity, which in my experience in talking with elementary school teachers, is intriguing to every one of them. None has ever raised the question about well, this is a highly artificial situation. What they centered on was the thinking that's involved and it came in a natural way. That's the value of it. I should mention that in the Benezet study, when students in the experimental group had this problem posed to them, they made the solution far more complex because they couldn't believe that Benezet was asking them such a simple question. In the second of Benezet's three articles, he gives an outline of the syllabus for the experimental group. And there are some interesting ideas there. Working with numbers is clearly de-emphasized. Working with large numbers practically doesn't occur, but he outlines the emphasis placed on thinking.

Sawada: You have a question?

Shimada: May I use the overhead please? Professor Becker made reference to key words in his talk. The key word approach is often used in a wrong way. The role of key words is to help recall a typical situation, and should not be directly connected with the kind of operation to be used. By key words one can imagine a situation, and transform its image to a familiar model situation represented by such semi-concrete materials such as magnetic buttons, marbles, tape, and so on, and often then decide which mathematical operation should be applied by that model. Thus, the key word approach may be useful if such steps as

key words ⟶ situation ⟶ model ⟶ operation

are followed in this proper order. However, if key words are directly connected only to the final step by shortcut, that is, cutting off the intermediate steps, they will likely cause wrong results when the combination of givens and required is different from the standard ones.

Becker: I think your observation is a very good one and I agree that this is the part in which we don't engage kids. I would say that people like Whitney, Easley and many others would agree and that what happens in the process is that kids, for a variety of reasons, want to satisfy the teacher, to get an answer, to get something. So they cut off the other important steps.

Shimada: If a teacher suspects that his children are using key words in a wrong way, I would like to suggest that the teacher ask the children to represent the situation by a model using magnetic buttons, marbles, tapes, etc.

Becker: I have a couple of quotes from Easley which are very much in the spirit of what Professor Shimada says. For example, Easley emphasizes giving difficult problems to kids within the first few weeks of first grade, which are thinking problms and which do not provide the opportunity for the children to focus on "doing something" with the numbers. Let me read one short quotation. After identifying a problem worked on by children in first grade, within the first few weeks of school, he says "perhaps this shows children early that there may not be any useful rules to guide the decision when to add, subtract, multiply or divide and when to use more than one operation." In other words the emphasis is on thinking.

Inouye: So early on the child's experience, even in the first grade, the student is given the idea that there may be no easy rules to decide what operation because the problems are of sufficient complexity that there may be no easy rules, because it doesn't involve numbers.

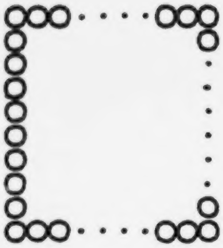
Becker: There is another quotation that I think might be quite useful, but we are running a little short of time.

Nohda: This point that came up in our discussion about the unrealistic aspect or how unreal some of these problems are in relation to the actual everyday life, the real world, has been pointed out as early as the 1930's in Japan in a famous work commonly known as the Green Book. In it, the relationships between the reality of the real world versus the fictional, the imaginative and the fantastic are discussed.

Sawada: Are there other comments or questions?

Becker: Here is a problem that might be of interest.

How many marbles are needed to make the boundary of a square of marbles, with ten marbles on each side?



(Note to the reader: This problem and different approaches to its solution were discussed at some length.)

Miwa: Thank you, Professor Becker. You cited my problem, Example 8 on page 372. This is a very popular problem in textbooks of elementary school in Japan. In order to solve this problem, Japanese students are encouraged to use an inductive way, i.e., though the number of a side is 10, the student may vary the number of a side from 3, to 4, and so on, make a table, and find a rule for this situation. Then the student applies the rule to original problem. In order to solve the problem, the student need only to be able to draw the correct figure and then count. But this is not the case, the point is to let students consider the problem inductively.

Becker: With sixth, seventh and eighth grade students, after I used the problem with entire classes of students and after some students discovered that they counted the corners twice, they very quickly picked up the general formula in an inductive manner; then when I switched to a triangle or a pentagon they (1) got very excited about it and (2) could get the solutions very quickly.

Silver: As a follow up to that comment, I have had similar sorts of experiences with this kind of problem. It then becomes an interesting exercise for students to verify that each of those patterns leads to a slightly different formulation of the same general rule and then becomes an interesting exercise to verify that they all are, in fact, equivalent to each other. A nice algebraic exercise is provided and it illustrates one way in which problem solving can be used and integrated with the normal goals of the curriculum. Whereas, in the past teachers might have given students each of those expressions and asked them to show that they were equivalent. Such a task would not be nearly as interesting as it is when it arises out of this problem situation. This is an illustration of the point that was being raised this morning about how do you find time to teach problem solving along with all the other skills. They can be integrated together.

Nohda: In the case of Japan, if you present students in first grade and above the problem on page 372, the Japanese students suspect that the teacher must be setting up a trap. In the case of the American student, do they show that?

Becker: In my experience, the students just go to work on it. And they also have a large feeling of satisfaction when they (wrongly) see that the answer must be 40. But there was real excitement when they discovered, in the case of the triangle, that they had counted each of the corners twice.

Here is another problem that may be of interest:

> On a farm there are 10 animals - cows and geese. Altogether they have 34 legs. How many cows and how many geese are on the farm?

Very frequently teachers will apply an algebraic approach and comment that if you propose this problem for elementary school students, they don't have the algebraic skills. It might be that elementary school students will focus on ten heads represented by circles, each with two legs and then ask, "How many more legs are there?" There are 14, so add them, two at a time, to the circles, and the solution is there. So as teachers, we discuss this solution, an algebraic one, a computer solution and still one or two others. Then we discuss which is the best solution. And I argue that this is the best solution even from the point of view of a professional mathematician. But frequently teachers will not agree.

Sawada: Only one more question before we close.

Hashimoto: I have one question, in your conclusion on page 379 and 380 you quoted Professor Easley's observations. Concerning #1, 2 and 4, I think they are possible for American classroom teachers, but I think #3 has a strong cultural aspect because it requires cooperation among teachers. How do you feel about this?

Becker: Yes, there could be some problems. On the other hand, Easley has formed a group called DIME (Dialogues in Mathematics Education) which involves elementary school teachers of mathematics with whom he is trying to apply the ideas outlined in his book - he is trying to implement these ideas in the classrooms. One important aspect that he has found is for a teacher, in order to implement what he proposes, to get another teacher who thinks along the same lines and who will cooperate. It takes more than one teacher to implement the ideas, both in terms of practical aspects and in terms of psychological support because the teaching situation is so different. Also, a teacher has his or her classroom and a certain amount of freedom to organize that group of students the way he or she would like, and so can try this new approach. Easley has had some success at this.

Sawada: Thank you very much, our time is now gone.

Discussion of Formulation of the Seminar Report

Becker: Professor Edwin McClintock has joined us. He is teaching here at the University of Hawaii this summer. The materials he is using are the SCOPE materials from Sid Rachlin's project. And since he was on campus, we have asked him to feel free to come over and sit in on our sessions.

McClintock: Thank you. I am enjoying it.

Travis: This afternoon discussion will center around the formulation of the seminar report. First of all, Professors Becker and Miwa will make preliminary remarks and then we can get into the discussion.

Becker: Professor Miwa and I have spoken about this with others and we think we would like the session to be a very free and open one in which all of us should share our ideas about preparation of the proceedings. Then, also, we want ideas on which we might follow up in terms of research after the seminar is completed, as we go into the next phase of this whole process. We invite and encourage all of you to share whatever ideas you have in a very open atmosphere. We will take notes and then we can pull some of these ideas together this evening so that the discussion tomorrow can focus more specifically on some of them.

Miwa: Of course, as Dr. Becker has said, please be free with your comments and suggestions and let us focus on areas such as what went on in the seminar and what will come out of it in terms of projects. A second point, of course, there is no way that we can solve all of the problems that have been brought up, but it's important to follow through on what issues have been left not addressed or what has not been touched upon. The third point is to investigate what approaches to take toward problem areas or problem points and how Japan and the United States may through this group carry on joint research, in what shape and form, how to follow through on these areas, and what problem areas may still exist. Although there is a physical distance between Japan and America, there are various means of communications that are available to us and that should not hold anyone or anything back from forming a joint research group. Some of us have expressed ourselves at this conference more than others, and some have not really stated too much in way of their opinions. But now there is an opportunity for all to speak. Now, please, all of you feel free to throw anything out to us.

Travis: Any responses?

Nohda: We have come together to make an effort at least to understand each other's problems and work together. There may be a need for a common language or what issues or certain terminology are necessary for common understanding; for example, what is problem solving? What does it mean when we use words such as key words? Though there are necessarily differences between Japan and the U.S., we can also look for some common grounds that we agree on and can build on. There are many merits to a seminar such as this and, of course, relating to what I stated earlier, about trying to approach the same problems both from the American side and from the Japanese side. We need to establish common areas as well as identify the differences and to note the characteristics of each; that is, how each side approaches a particular point, whether this takes the shape of curriculum research or some other such project.

Silver: I'd like to make a few suggested topics just to get them on the floor for discussion. These are things that I've noted in the last few days' discussion that seemed to come up fairly frequently. I'll formulate them rather briefly and then perhaps they will be topics for more discussion. One theme that I think has been sounded quite frequently by the U.S. delegation is an interest in studying more clearly the teacher preparation, teacher education programs in both countries, especially with respect to the teaching of problem solving and how to promote mathematical thinking. A second topic that was mentioned several times in the first few days, but not recently, is the interest in studying the methods and techniques used by exceptional teachers of problem solving in both countries. There are some teachers in each country who can be identified as being exceptional, good teachers of problem solving, and a comparison of the methods used in each country might be of substantial interest. The third and final comment or area is one that would involve taking a set of problems which we could agree upon or interesting problems for our students, finding appropriate grade levels at which to study the solution of those problems and then examining in some detail the processes used by American students and Japanese students as they solve those problems.

Ishida: In Japan, when studying the situation concerning problem solving, although I realized that problem solving was a large issue, I considered that there were considerable differences about problem solving that is taught and points being emphasized in problem solving. Having attended this seminar and now knowing about the problems and strategies which are being treated in the U.S., I now realize that the situations and the whys and the hows are not so different between our two countries.

But I would like more discussion about the basic points such as "What is problem solving?" and "Why do we teach problem solving?" For example, do we teach problem solving for cultivating students' mathematical thinking, for developing abilities to solve real problems or for attracting a lot of attention to mathematics?

Becker: One way in which we learn a lot about another country's mathematics curriculum is to actually look at the books that are used in the classroom and the books that are used in teacher training. I wonder whether it would be feasible for us to attempt to get translations of some of the Japanese educational materials into English for our use, and the other way also?

Shimada: While there exists an issue of trade imbalance between the U.S. and Japan, with an excess of export on the Japanese side, regrettably it is a fact that there exists a similar situation in a reverse direction in exchanging information on mathematics teaching between the two countries. We have been importing much from U.S. and other countries but not exporting our products in mathematics education abroad.

It would take too much time and energy for us to translate literally our textbooks into English. Having somebody else do it might not turn out as well as we would hope, because of a host of other possible problems that would come up about translating. If you would be satisfied with rough translations of those portions which deal with problem solving, it might be possible for us to do it through our cooperative efforts - that is just one possibility.

Clarkson: A more reasonable approach might be to decide what information we want from the Japanese textbooks and from the American textbooks and then we could analyze our textbooks and the Japanese could analyze theirs in terms of those particular characteristics. Then we would be able to compare those results. That probably would be better than attempting to do the translations and then either work on them all here or work on them all in Japan.

Travis: Would it be more useful, instead of looking at textbooks, to look at test materials? The textbook really just provides an available curriculum and many times a test will better reflect what was actually presented in the classroom and what is actually implied for the student to learn.

Kantowski: I would like to expand on what Dr. Travis has just said. Another area that we need to look at is that of evaluation and how evaluation of problem solving is done in each country. We might take a topic such as pattern finding and look at how are these things evaluated in each country. In one of our earlier days, the whole notion of evaluation was brought up and we discussed how testing permeates our curriculum. I think it would be interesting to see if the same situation existed in the Japanese curriculum. There is also another issue, and I don't exactly know how to phrase it, much less to approach it, and that is that we have each brought our interpretations of what we do in our schools and in our country to the people from the other country. We, in turn, interpret what the other delegation has said in the light of our own experience. What I would like to see is some way, and I am not sure how to do it for the people in our delegation, to actually look at the materials and actually do the research. The problem is that there is a language barrier, but I would like to be able to find some way for us to be interpreting in our light some of the things that we see or can observe in the Japanese schools in your problem solving, in your students, etc. It's probably easier for the Japanese because their command of our language is so much better than our command of theirs.

Nohda: This relates to what Professor Silver was saying earlier, especially in the area of novice teachers. It is important that they actually see the teaching going on in the classroom, how it is done here, how it's done there. As Professor Hashimoto, Dr. Rachlin, and others have shown bits and pieces through video and then slides and so on, that would be most helpful in everyone seeing how teachers teach problem solving, how students study problem solving, to actually see through visuals what goes on in the classroom.

Rachlin: I'd like to react to that. I agree with Professor Nohda that it would be very valuable to be able to have videotapes. Some people came and asked me yesterday about whether we had videotapes of our classes. We don't have videotapes, but to have them to be able to share would help to bring those things to light that we were trying to talk about. Following up on what Professor Silver suggested earlier, his second suggestion had to do with looking at the classes of exceptional teachers and the ways that they teach. To videotape some classes and have them shared might be a way that we can both look and then be able to come together and talk about the things that we have seen. I think that that's different than my looking at someone teach a class, writing it down and sending it to the Japanese saying, "This is what I saw happen in the class." I am afraid the same thing may be true when we get to textbooks or other things. Unless we can see things in their original it would be hard to interpret whether you would see the same things as you looked at it. I found it valuable when we looked at the textbooks to see the English written right on the page where it was in the problem and the picture was there and you had the whole feeling of what was going on in that text.

Wilson: I think we started listing some possible topics for joint work and exchange and I would like to emphasize the exchange for a moment. I am not arguing against joint work but I think one of the things that is listed was a joint study group. I viewed that as one not necessarily where we were all working on the same research in two different sites but more research in the same areas about which we exchange information. That is one point that I make here. Maybe I can restate one of Mr. Ishida's comments. I think he was talking about an analysis of the goals of problem solving and the analysis of the goals of mathematics education and how problem solving fits into that. I think we need to continually have a reassessment of that as well as the procedures and techniques. I think that as time goes on and when we have resources or interests that overlap, we should sit down with each other's textbooks and spend a lot of time looking at them so that knowledgeable people in math education can interact about them. I don't know the mechanism by which that can be accomplished. However, Professor Miwa and Professor Nohda are sharing a student with us for a year. Maybe that student and some of my people could look at textbooks together. That's the kind of thing that I'm talking about. Whenever there is an opportunity we ought to follow up on it.

Shimada: It would be possible for us to sit down at the same table to compare textbooks, as Professor Wilson has suggested, because the supporting organizations of this seminar have a program for funding mutual exchange of researchers for a short-time stay in the other's country. But, of course, in order to be accepted, the proposal of the program must be well prepared so it can be evaluated.

Miwa: JSPS and NSF, which are the co-sponsoring organizations of this seminar, have the program of the United States-Japan Science Cooperation. But there are so many applications for support that it may not be easy for us to get approval for our application. So the plan must be deliberately designed and the goal must be clear.

Becker: The idea of doing joint research with children appeals to me. This would involve identifying the problems that we would use with Japanese and American children and the age levels. It might also require that when the data are being collected in Japan, there be an American or Americans there working collaboratively with the Japanese; and when it's happening in the U.S., that there be a person or persons from Japan working collaboratively with the person in the U.S. Concerning funding, it isn't always easy to get support but, on the other hand, in the U.S. there is some interest already by some agencies to look at a proposal that we prepare. That is, they are interested in this seminar as well as the proposals that come out of it. I am speaking of both governmental and private foundations, including Japanese foundations in the U.S. I think there are some other ways that we can interact professionally. In each of our countries there are research organizations and professional societies that have regularly scheduled professional meetings, and it might be good for reports to be given in both countries jointly by Japanese and American researchers and math educators. Another thought that comes to mind is the possibility of a teacher exchange. I know of several junior high school or secondary mathematics teachers in Japan who speak very good English who would be interested, I believe, in coming to the U.S. I am not so aware of American teachers who can handle Japanese but, on the other hand, we haven't looked yet. My guess is that there are such individuals in the U.S. and maybe this kind of an exchange could be very valuable on both sides for reporting the perceptions of each of what actually goes on in the classroom situation. I would also like to suggest the possibility that at some time in the future - three years, five years, seven years - as research goes on in Japan and in the U.S. and perhaps cooperatively, that a forum such as this would be appropriate to exchange information on what has been learned about problem solving behaviors of children and teaching problem solving.

Shimada: I would like to add one point to support Professor Becker's suggestion. It seems important to set up a network for exchange of information between the two countries in order to facilitate a wide and quick dissemination of information from this Seminar to those interested who are not in this group.

For that purpose, it would be better to appoint one person on each side to serve as a channel through which information from one side is to be distributed within the other side. Otherwise much more money would be needed for communication because of postage.

Kantowski: We have been making suggestions about the kinds of things that we might be able to do. The notion of funding from the various organizations has come up and both Profesors Miwa and Becker said there is some interest in it. But the ideas have to be very specific. Perhaps if we have an idea that we are interested in, we could outline it specifically and give it to Dr. Becker or Dr. Miwa so that if they are thinking about putting something together they have some specific ideas that they might want to follow up.

Nagasaki: I am now thinking not only of future tasks but also the findings of this Seminar. But, now, I will talk about the findings. I think, generally speaking, we must seek differences and commonalities between the U.S. and Japan. I find fifteen differences and four commonalities as below. First of all, I will enumerate the differences with some explanation and present them as a dichotomy, the former for Japan and the latter for the U.S. Of fifteen differences, first, the relation between school teachers and university teachers: together or separately; second, the place from which research is started: from classroom or from researcher's room; third, research method: paper-pencil centered or thinking aloud centered; fourth, research subject: elementary students or high school students; fifth, the meaning of problem solving: the way of mathematical thinking or solving problems; sixth, problems in problem solving; mathematically simpler problems or mathematically sophsiticated problems; seventh, the content of problem solving: process oriented or strategy oriented; eighth, the treatment of keywords in word problems: indirectly or directly; ninth, teaching computation: to have written mathematical expression carefully or to have only the result written; tenth, teacher's activities in classroom: children centered or teacher centered; eleventh, children's thinking aspects in a classroom: various aspects in a homogeneous class or a few aspects in a heterogeneous class; twelfth, several issues on textbooks: centralized or decentralized, etc; thirteenth, parents' expectations: stronger or moderate; fourteenth, the cause of uniformity: entrance examinations or standardized tests; fifteenth, the last point, social atmosphere: toward a group or toward an individual. Next, I talk about four commonalities. First, all of us think that problem solving is very important; second, we agree that children cannot obtain good achievement in linguistically complex problems; third, we recognize that real world problems are important; fourth, we also agree that teacher education plays a vital role in problem solving. These are my observations.

Travis: Our time has passed quickly, too quickly, and now we will have to stop. There will be more opportunity tomorrow to follow up on this discussion. Thank you all very much.

Professor Miwa's Paper

Rachlin: In this session we'll be talking about mathematical model making in problem solving. The speaker for this morning is Professor Miwa and through the week we have had an opportunity to have many interactions with Professor Miwa. There are some things that you have already learned through the week. One of them is his interest in mathematical model making and in the process of mathematization of situations that come from outside of mathematics. Something that you might not know and one of the things that pleases me to be the person to serve as Presider is that Professor Miwa also worked at a laboratory school, at the laboratory school associated with the Tokyo University of Education. In this morning's session then we will be talking again about mathematical model making and I'd like to present to you Professor Miwa.

MATHEMATICAL MODEL-MAKING IN PROBLEM SOLVING--
JAPANESE PUPILS' PERFORMANCE AND AWARENESS OF ASSUMPTIONS

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1. Introduction

The term "mathematical model-making (MMM)" is often used in the process of application of mathematics. Typically the process consists of the following four steps: (1) Formulation, (2) Mathematical Works, (3) Interpretation and Evaluation, (4) Improvement of Model (Step (4) involves repeating steps (1) and (3) several times). Of the above four steps, the most important and difficult one is "Formulation" (step 1). Its essential element is to establish suitable assumptions based upon deliberate suppression or neglect of irrelevant details. In problem solving, the above steps are also applied. Thus, we can regard the application of mathematics as problem solving in the real world itself.

The first task in problem solving is to define or formulate the problem. For this, it is necessary to establish or construct and to limit, if need be, the field of discourse. Naturally, this should lead to the establishment of suitable assumptions. Establishing assumptions is necessary when solvers apply mathematical techniques, which they know well and can utilize freely, to the problem.

In the experimental teaching of MMM to upper secondary school pupils several years ago, the author noticed that some students experienced anxiety about MMM. This anxiety was caused by the concern that the assumptions, which are required in MMM, might deprive the situation of its significance.

MMM is dependent on the establishment of assumptions, and, even in elementary school arithmetic, the assumptions are taken for granted implicitly or unconsciously. In this paper, the author would like to consider mathematical model-making and awareness of assumptions, to clarify the extent to which Japanese pupils are aware of the assumptions in problem solving and then to describe the implications of this for teaching mathematical problem solving.

2. Mathematical Model-Making and Awareness of Assumptions

In this section, we will consider the meaning and significance of mathematical model-making, and awareness of assumptions in model-making.

2.1 Mathematical Model-Making

The term "mathematical model-making (MMM)" is often used in the process of application of mathematics. Recently, mathematics has been applied to very divergent areas in diverse ways. The common feature of these applications of mathematics is said to be the application process itself, which may be called the MMM process (Miwa, 1984a). The MMM process typically consists of the following four steps, which rest on the recognition by model-makers that the situation under consideration is of significance to be understood or to be solved:

(1) to formulate the situation into a problem in mathematical terms (Formulation),
(2) to find out mathematical results (Mathematical Works),
(3) to interpret and evaluate the results with respect to the original situation (Interpretation and Evaluation),
(4) to improve the model to get better results (Improvement of Model).

This process can be illustrated as in the diagram (Figure 1). The need to go around the loop several times for improvement of the model is implicit in the diagram.

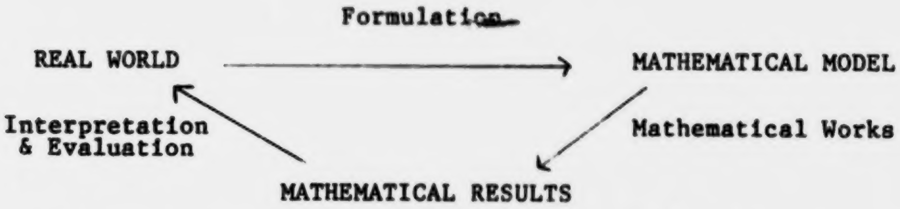


Figure 1 The Process of Mathematical Model-Making

Of these four steps, the most important and difficult one is "Formulation" (step 1). This step involves setting up suitable assumptions based upon deliberate suppression or neglect of irrelevant details, identification of the questions, and representing them in mathematical symbols (Miwa, 1984a). However, the importance of "Improvement of model" (step 4) should not be forgotten.

In problem solving, the above steps are also applied. Thus, we can regard the application of mathematics as problem solving in the real world itself. In fact, the first task in problem solving is to define or formulate the problem. This is step (1) of the above process. And for this, it is necessary to establish or construct and to limit, if need be, the field of discourse. Of course, steps (2)-(4) are also important in problem solving.

Next, we will consider the significance of MMM in education. First, as was shown above, MMM is primarily concerned with the application of mathematics. Today it has become increasingly recognized that the application of mathematics is an indispensable component of mathematical education. Consequently, it is essential that school mathematics be made more applicable (Miwa, 1984a).

Second, MMM is concerned with tackling real situations. Therefore, it increases the motivation of pupils above the level of mere intellectual curiosity and lets pupils participate in the developing process of knowledge (Miwa, 1984a).

Third, but not least, MMM contributes to the learning of axiomatics in school mathematics. In fact, Formulation (step 1) necessitates the postulation of some fundamental propositions, and the mathematical model thus formulated is treated just as an axiom in the problem. Then, by utilizing deductive reasoning based upon the axiom, results are obtained. Thus, both the results and their validity are dependent upon the initial mathematical model. Therefore, if the results are unsatisfactory in the real world, the initial model itself needs to be improved. That is, model-making connects the empirical and real world and the transcendental world of deduction (Shimada, 1977, 1985).

In Japanese school mathematics, application of mathematics is not emphasized very much. At present, application of mathematics does not exist as an independent subject of mathematics in the Course of Study for upper secondary school nor at the lower secondary and elementary school levels. Instead, applications within several mathematical topics are taught; for example, in the application of equations, geometry, calculus, etc. In addition, it should be noted that upper secondary school pupils in Japan seem to have little interest and less confidence in the application of mathematics in the real world (Miwa, 1984a).

Thus MMM as a process of application of mathematics has not been pursued with much interest in reference to Japanese school mathematics. However, in mathematical problem solving, this is not the case. MMM is often one of the more important issues stressed in Japanese school mathematics (Miwa, 1984a). In the next section we notice the importance of awareness of assumptions in MMM, and will concentrate on it in the succeeding sections.

2.2 Mathematical model-making and awareness of assumptions

As we have described in 2.1, to set up suitable assumptions based upon deliberate suppression or neglect of irrelevant details, to identify the questions and to represent them in mathematical symbols are all components of "Formulation" (step 1). In problem solving the first task is to define or formulate the problem. For this, it is necessary to establish or construct and to limit, if necessary, the field of discourse. Naturally, this should lead to the establishment of suitable assumptions. Establishing assumptions is necessary when solvers apply mathematical techniques, which they know well and can utilize freely, to the problem.

In the experimental teaching of MMM to upper secondary school pupils several years ago, the author noticed that some students experienced anxiety. They said that there were so many factors to be taken into consideration that real situations were beyond their reach, and they were concerned that making bold assumptions might deprive the real situations of their significance. Thus they could not appreciate the value of MMM itself (Miwa, 1984a).

MMM is dependent on the establishment of assumptions. In addition, it should be noted that even in elementary school arithmetic, assumptions are taken for granted implicitly or unconsciously. In arithmetic, we often make assumptions necessary for getting mathematically useful results, as in the case of typical word problems. Examples will be given in the next section. Typical word problems deal with what appear to be fully mathematized situations, but the assumptions necessary for mathematization can only be recognized by pupils if teachers make them explicit and if pupils exercise sufficient care (Miwa, 1984b).

Next, we would like to describe how aware Japanese pupils are of assumptions in mathematical problem solving.

3. Survey of Japanese Pupils' Awareness of Assumptions

In this section, the results of a survey on pupils' awareness of assumptions in mathematical problem solving will be discussed.

3.1 Method of Survey

The following two problems were posed:

(1) Mr. A accomplishes a certain work in twenty days, and Mr. B does this work in thirty days. If A and B do this work together, how long would it take them?

(2) Mr. A goes around a pond in twenty minutes, and Mr. B goes around this pond in thirty minutes. If they start at the same point simultaneously but go in opposite directions, when would A and B meet again?

Three questions were asked about these two problems:

(i) Solve both problems.

(ii) Write down the assumptions you used when you solved the problems, assuming that both these problems are situations in the real world.

(iii) If the problems are assumed to be situations in the real world, do you think your solutions are good or not (for lower secondary school pupils only)?

These questions were asked of about 280 pupils from the Elementary School (sixth grade), the Lower Secondary School and the Upper Secondary School attached to University of Tsukuba (UT) (about forty pupils in each grade), and of about forty students at UT who intend to become secondary school mathematics teachers. We should mention that all the pupils and students surveyed entered their respective schools or university by passing difficult entrance examinations. Thus, it is assumed that they are more able and more willing to study mathematics than most pupils of public schools. The sample was confined to these pupils and students because we were afraid that the above two problems, while typical word problems in arithmetic, would be too difficult for average public school pupils to solve. Hence it may be difficult for the latter category of pupils to provide us with useful information about their awareness of assumptions.

3.2 Results

In the following sections, problem (1) will be referred to as the 'work problem' and problem (2) as the 'walk problem.' The results will be shown in the order of the questions (i), (ii) and (iii).

(i) Solving the problems

The results are summarized in Table 1. Numbers in the table represent the number of pupils who gave correct answers, and numbers in parentheses stand for the percentage of those pupils.

Table 1 Numbers Of Correct Answers To The Problems

School Level	Grade	Total Number	Walk Problem	Work Problem
Ele. Sch.	6th	37	25 (68%)	29 (78%)
Lower Secondary School	7th	41	40 (98%)	39 (95%)
	8th	39	33 (85%)	31 (79%)
	9th	38	38 (100%)	38 (100%)
Upper Secondary School	10th	43	36 (84%)	34 (79%)
	11th	43	41 (95%)	41 (95%)
	12th	34	34 (100%)	33 (97%)
University		44	42 (95%)	39 (89%)
Mean Percent			91%	89%

Some of the methods used by the pupils to solve the problems are shown below:

(a) The method used by most pupils for both problems was·

$$1/(1/20 + 1/30)$$

(b) Examples of other methods are:

*walk problem. One pupil assumed that the perimeter of the pond would be 60 m. "As Mr. A walks around the pond in 20 minutes, he walks 3 m. per minute. Mr. B walks 2 m. per minute. Thus, A and B would approach each other at the rate of 5 m. per minute, and would meet in 12 minutes because 60/5 = 12."

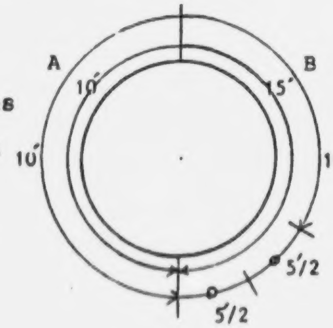
The characteristic of this method is to assume freely the perimeter of the pond. This method was often used by pupils in 6th grade. In fact, ten pupils (40% of those who gave correct answers) solved the walk problem using this method. This method can also be applied to the work problem, but only three pupils (10% of those who gave correct answers) solved the work problem using this method. On the other hand, a few pupils in 7th grade and above utilized this method.

*work problem. One pupil expressed the quantity of work as x. "As Mr. A accomplishes the work in 20 days, he does x/20 a day. B does x/30 a day. Thus, A and B do x/20 + x/30 a day together, and they accomplish the work in x/(x/20 + x/30) = 12 (days)."

The characteristic of this method is to use algebraic expressions. This method was rarely used by lower secondary school pupils, but a considerable number of upper secondary school pupils and university students solved the problem using this method (eleven pupils in 10th grade (32%) used this method).

We would also like to draw particular attention to incorrect answers given by pupils. Here, typical examples of the way in which the problems were solved incorrectly will be focused on. This, we believe, is of much value for finding the difficulties in mathematical problem solving.

*walk problem In 10 minutes A walks half of the perimeter of the pond, and in 15 minutes B walks half of the perimeter also. In 10 minutes B walks a third. As can be seen in the figure on the right, 5 minutes overlap. Therefore, 5 is divided into two equal parts and 5/2 + 10 = 12.5 is the answer. (6th grade)



*walk problem (20 + 30)/2 = 25 is the answer. This is the average of 20 and 30 (6th, 8th and 10th grades).

*work problem (20 + 30)/2 = 25. This is the average of 20 and 30. As A accomplishes the work in 20 days alone, 25 days is rather long and 25 must be divided by 2. Therefore, 25/2 = 12.5 is the answer (6th, 8th and 10th grades).

In the methods outlined above which gave rise to incorrect answers, an "averaging strategy" was evident. That is, as speed is constantly changing, only average speed can be calculated and, therefore, the reasoning was that problems involving speed should be solved using the concept of average. However, as they knew only arithmetical mean, they ended up with incorrect answers.

We summarized the results for question (i) as follows:

a. Most pupils from the sixth grade of elementary school through to university students could solve both the walk problem and the work problem. In fact, the mean percent of correct answers (MPCA) for both problems was about 90% for the seventh grade and above. However, MPCA was a bit lower (76%) for the sixth grade.

b. For secondary school pupils and university students, MPCA for the walk problem was higher than or equal to that for the work problem, but for sixth graders, MPCA for the work problem was higher.

c. A few pupils and students could not solve the problems correctly. An "averaging strategy" was evident in the methods which gave rise to the incorrect answers.

(ii) Writing down the assumptions

We classified the writing of pupils and found typical examples of the assumptions of which pupils were aware. In the following tables, we focus on these typical examples. Tables 2 and 3 illustrate the numbers of pupils who were aware of the typical assumptions, irrespective of whether their solutions were correct or not.

walk problem

In Table 2, "uniformity of walking" refers to the speed of Mr. A and B. For example, A and B walk uniformly around the pond; i.e., they both walk at a constant speed. "State of circumference" refers to the state of the path around the pond along which A and B walk. For example, the path around the pond is uniformly even or flat. This assumption might be due to the fact that as A and B go in opposite directions the state of the path must be the same and even in both directions. In contrast to these two assumptions, "method of solution" refers only to the assumptions used as a tool in the solution. A typical example is that the length of the circumference is assumed to be 1. As mentioned previously, one 6th grade pupil assumed the length to be 60 m.

Table 2. Number Of Pupils Who Were Aware Of The Assumptions (walk problem)

School Level	Grade	Total number	Uniformity of walking	State of circumference	Method of solution	No answer
ES	6th	37	21 (57%)	1 (3%)	11 (30%)	1 (1)
LSS	7th	41	35 (85%)	2 (5%)	3 (7%)	2 (0)
	8th	39	30 (77%)	5 (13%)	2 (5%)	3 (1)
	9th	38	34 (89%)	7 (18%)	1 (3%)	1 (0)
USS	10th	43	31 (72%)	7 (16%)	3 (7%)	2 (1)
	11th	43	36 (84%)	21 (49%)	0 (0%)	2 (0)
	12th	34	31 (91%)	6 (18%)	3 (9%)	0 (0)
Uni.	-	44	32 (73%)	6 (14%)	3 (7%)	3 (1)
Mean Percent		—	79%	17%	9%	

(In this table, numbers in parentheses in the "No answer" column represent the numbers of pupils who did not respond to this question and gave either incorrect answers or no answer for the walk problem.)

Also, several incidents of miscellaneous writing were found. Some were comments but not assumptions, as depicted by the following examples:

*When one goes around the pond, he must walk along the path at a constant distance away from the pond. This is very difficult to do, and naturally may result in an error of a minute or so in the time needed to go around the pond. (8th grade)

*A pond is not always circular and it is not always located in a flat area. Therefore, this problem seems to be not in the real world but in the imaginary world. Boldly speaking, in arithmetic the impossible is dealt with. (6th grade)

*When both A and B catch sight of each other, their speeds will naturally increase. But this is not assumed. What is assumed is that A and B will walk like machines - that is, mechanically. (8th grade)

*One does not necessarily go around a pond in a constant and fixed time. Speed over 1 km. will vary from time to time. Therefore the average speed is given here. I will solve the problem by utilizing the average speed of A and B. (6th grade)

work problem

Table 3 illustrates the results for the work problem.

Table 3. Numbers Of Pupils Who Were Aware Of The Assumptions (work problem)

School Level	Grade	Total number	Uniformity of work	Degree of independence	Method of solution	No answer
ES	6th	37	23 (62%)	1 (3%)	8 (22%)	3 (1)
	7th	41	31 (76%)	8 (20%)	2 (5%)	1 (0)
LSS	8th	39	28 (72%)	9 (23%)	2 (5%)	3 (3)
	9th	38	24 (63%)	18 (47%)	0 (0%)	1 (0)
	10th	43	31 (72%)	14 (33%)	5 (12%)	5 (2)
USS	11th	43	32 (74%)	27 (63%)	0 (0%)	0 (0)
	12th	34	27 (79%)	20 (59%)	1 (3%)	0 (0)
Uni.	-	44	27 (61%)	15 (34%)	3 (7%)	2 (2)
Mean Percent			70%	35%	7%	

(In this table, numbers in parentheses in the "No answer" column represent the numbers of pupils who did not respond to this question and gave either incorrect answers or no answer for work problem.)

In the above table, "uniformity of working" refers to the working rates of Mr. A and B. For example, A and B work uniformly like machines or robots without taking a rest or becoming sick; i.e., at every moment, both A and B work at the same rate. "Degree of independence" refers to the degree of cooperation or disturbance between A and B in their work. A typical example is that the working rate of A when working together with B is the same as when he works alone and vice versa. Underlying this, are the assumptions that if A and B work on a cooperative basis, then they could accomplish the work in less time, and if they do not cooperate, then they would disturb each other and the work would require more time to finish. In contrast to these two assumptions, "method of solution" refers only to the assumptions used as a tool in the solution.

A typical example was that the amount of work to be done was assumed to be 1. For another example, one 6th grade pupil assumed that 600 windows had to be cleaned.

Again, several examples of miscellaneous writing were found:

*Work should be quantified. That is, it should be expressed in numbers and be divided into equal parts. (12th grade)

*We must consider the difference in wages between A and B. (10th grade)

*We assume a restriction on available tools in the work. For example, if A uses a truck and accomplishes the work in 20 days and B uses a truck and completes the work in 30 days and if there is only one truck available, then the work cannot be accomplished in the calculated 12 days. (11th grade)

*If we round off 19 days and 3 hours, it becomes 20 days. In the same way, 29 days and 8 hours, becomes 30 days. Therefore, the actual length of 20 days or 30 days needs to be clarified. (8th grade)

We summarized the results for question (ii) as follows·

a. For both the walk problem and the work problem, most pupils were aware of the need to assume uniformity; for example, in the walk problem that Mr. A and Mr. B go around the pond at a constant rate. The percentage of awareness of this uniformity was higher in the walk problem than in the work problem, except for sixth grade pupils.

b. In contrast to the uniformity assumption, in the work problem, a few pupils below the eighth grade were aware of the degree of cooperation. That is, they were aware of the assumption that the work per day done by the two persons is the sum of the work done per day by A and by B. The number of pupils who were aware of this assumption increased as the grade level increased.

In the walk problem, fewer pupils were aware of the state of circumference. However, the percentage of tenth grade pupils who noted this assumption was exceptionally high. The reason for this is unknown.

c. Assumptions for the method of solution were limited mainly to sixth grade pupils, with very few pupils from other grades being aware of them. This may be partially because sixth grade pupils interpreted assumptions for the problem in different ways from other grade pupils.

d. Some pupils commented that both problems were of a mathematical type and thus did not constitute problems of the real world and, therefore, it was of little value to solve these problems. This can be seen more clearly in the following section (iii).

e. As for the pupils who gave either incorrect answers or no answer (PIA), the following were noticed:

*PIA were aware of the assumption of uniformity to the same extent as the pupils who gave correct answers (PCA). Many PIA utilized the "averaging strategy" in solving both problems and this would correlate to their awareness of uniformity.

*In the work problem, there was considerable overlap between PIA and pupils who did not respond, as shown in Table 3. This may imply that being unaware of assumptions accompanies being unable to solve problems.

(iii) Determining whether the solutions are good or not.

Table 4 illustrates the numbers of pupils who responded whether they thought their solutions were good or not, irrespective of whether their solutions were correct or not.

Table 4. Response Of Pupils Concerning Their Solutions
(Lower secondary school)

Grade Level	Total	Walk problem		Work problem	
		Good	Not Good	Good	Not Good
7th	41	6 (15%)	35 (85%)	8 (20%)	33 (80%)
8th	39	7 (18%)	31 (79%)	1 (3%)	38 (97%)
9th	38	5 (13%)	33 (87%)	2 (5%)	36 (95%)
Mean percent		15%	84%	8%	91%

(In this question, one pupil in 8th grade did not respond for the walk problem.)

Some responses of the pupils are given below:

*The answer "12 minutes" presents no problem, as it has not got anything to do with other people. (8th grade)

*Except for the assumption of uniform speed of walking, because the solution is gained by using a long-trusted method of computation, namely, tabibito zan (travellers computation), the solution must also be equally reliable. (7th and 8th grade)

*Work by hand is not done uniformly and therefore it is different from work done by machines. (9th grade)

*When we assume that the problem is in the real world, the days required must be more than those calculated. It is necessary to take such factors as the health of the person and the weather on the working days into consideration. (8th grade)

*If we consider the problem mathematically, the solution will be good. However, if we consider it physically--for example, one might take a rest or might work harder than usual--the number of days actually taken will not be the same as the number of days calculated. (8th grade)

*Assuming that a problem is in the real world, we cannot predict what happens. But it is of no use to say such things. I believe that these factors should be ignored and that persons with any common sense will work together and accomplish the job in approximately the numbers of days calculated. (9th grade)

*These problems are good for estimating things. (9th grade)

We summarized the results for question (iii) as follows:

a. Most lower secondary school pupils responded that they did not think that their solutions were good if the problems were assumed to be situations in the real world. The typical response was that the solution was only "mathematical" and fictitious. This reaction was most apparent in seventh and eighth graders. However, some ninth graders said that the problems and solutions were mathematically modeled.

b. For eighth and ninth grades, more pupils responded that they thought their solutions were better in regard to the walk problem than for the work problem. This is because it is easier to walk uniformly for 20 or 30 minutes than to work uniformly for 20 or 30 days, and going around a pond is an action which does not have anything to do with other people.

4. Implications For Teaching Problem Solving

In this section we will consider the implications of the results of our survey. First a provisional conclusion will be described, and then proposals for teaching problem solving will be given.

4.1 Provisional conclusion from the survey

From the results of the survey shown in 3, we can draw the following conclusions:

Most pupils in the sixth grade of elementary school and in the lower grades of lower secondary school were able to solve both the walk problem and work problem, and were aware of the assumption of the uniform rate of walking and working in the problems. In addition, most of the pupils who gave incorrect solutions used an "averaging strategy" for solving problems and were aware of the uniformity. However, most pupils, especially the eighth grade and below, were unaware of the degree of independence (work problem) and the state of the circumference (walk problem).

On the other hand, most pupils of these grades believed that mathematical problems belonged to the mathematical world and therefore mathematical problems could not be applied in the real world. Thus, they felt that mathematical solutions were irrelevant to the real world and they were not good.

This suggests that most pupils in the sixth grade of elementary school and in the lower grades of lower secondary school are partially aware of the assumptions in typical word problems but they only see them as assumptions in the mathematical world and cannot see their relevance to the real world. But at a certain point--at the ninth grade level or above--we may safely assume that pupils become aware of the significance and meaning of model, and they understand that the establishment of assumptions is essential for model-making and that the conclusions derived from these assumptions are of value.

4.2 Proposals for teaching the awareness of assumptions in problem solving

Based upon the above provisional conclusion, we would like to propose the following two points in regard to the pupils' awareness of assumptions in problem solving:

a. As shown above, pupils in the upper grades of elementary school and lower grades of lower secondary school believe that mathematics is apart from the real world. Therefore, if we do not make them aware of the assumptions and of the significance of mathematical models, they may have little confidence in the power of mathematics. Thus, attention should be focused on the pupils of upper grades of elementary school and lower grades of lower secondary school so as to make them reflect the above aspects of problem solving.

In addition, most pupils of these grades were aware of the assumption of uniformity of walking and working, and could use the "averaging strategy." This was probably due to the fact that the pupils at this level studied velocity and density, which are defined on the basis of uniform distribution, and accordingly were ready to be aware of the assumptions. This fact would make it easier to attempt the said proposal in the classroom, and if it is done, the pupils will gain an awareness of the above conclusion earlier, and become more confident in the utility of mathematics.

b. An effective way of teaching the awareness of assumptions is by having pupils reflect on the reason why a particular operation is used while they try to solve typical word problems. In this way, the pupils would be made aware of the assumptions about the operation.

For instance, in the work problem, the operation 1/20 gives us the work rate of Mr. A, or correctly speaking, his average rate of work per day. Here, it is assumed that A does the work at a uniform rate. The operation 1/20 + 1/30 in the same problem gives us the work rate per day of A and B when they work together. Here, it is assumed that A and B are working independently; i.e., the efficiency of both persons does not change by their working together.

At this point, pupils may raise objections like the responses shown in 3. Thus, it is a good opportunity for teachers to explain the meaning and significance of mathematical model-making.

5. Summary and Final Remarks

In this summary we will concentrate on the results of the survey on the awareness of assumptions in problem solving of Japanese pupils and their implications for teaching mathematical problem solving.

Two problems, a walk problem and a work problem, were posed and pupils from sixth to twelfth grades in the schools attached to University of Tsukuba (UT) and students of UT who intend to become secondary school mathematics teachers were asked three questions about each of these problems.

The results of the survey are summarized briefly as follows:

(1) Most of the pupils could solve both problems irrespective of their grades. A few gave incorrect answers, and the use of an "averaging strategy" was evident.

(2) In both problems, most pupils were aware of the uniformity assumptions. However, in the work problem a few pupils below the eighth grade were aware of the assumption of the independence of the two persons working, and fewer pupils were aware of the state of circumference in the walk problem.

(3) Most lower secondary school pupils responded that their solutions were not good if the problems were assumed to be situations in the real world. The typical response was that his solution was only "mathematical" and fictitious. This reaction was strongly evident amongst seventh and eighth graders. However, some ninth graders found that the problems and solutions were mathematically modeled.

We posed the provisional conclusion that for most pupils in the upper grades of elementary school and lower grades of lower secondary school, mathematical problems exist only in the mathematical world and they cannot be applied in the real world. But at a certain point--at the ninth grade level or above, we may safely assume that pupils become aware of the significance and meaning of a model. Therefore, we would like to propose the following two points in regard to the pupils' awareness of assumptions in problem solving:

a. Teaching awareness of assumptions should be focused on pupils in the upper grades of elementary school and lower grades of lower secondary school.

b. An effective way of teaching the awareness of assumptions is by having pupils reflect on the reason why a particular operation is used while they try to solve typical word problems. This will make pupils aware of the assumptions about the operation.

Finally we would like to point out some of the problems not dealt with by this study:

a. Survey of the awareness of assumptions in problem solving needs to be extended to average pupils of public schools. And, if necessary, the provisional conclusion should be revised.

b. The validity of the proposals for teaching problem solving needs to be examined.

c. The relationship of assumptions of which pupils were aware and pupils' solving strategies needs to be studied further.

d. Mathematical model making should be examined more broadly, and also, aspects, other than the awareness of assumptions, need to be investigated.

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Discussion of Professor Miwa's Paper

Rachlin: Thank you, Professor Miwa. Are there any questions or comments?

Kantowski: I would like to make one comment that reiterates something that we saw in many of the other presentations and that is that the notion of meaning is emphasized very much in the Japanese schools. One difficulty that we have is that when students get an incorrect answer they don't analyze the result and realize that their answer is incorrect. One thing that struck me was that when students got an incorrect answer and they realized that the answer was too large, they felt that they needed to do something to that answer to make it fit into what they estimated would be a reasonable answer. For example, in the problem in which one worker could do a task alone in 20 days and the other could do it alone in 30 days, a student who solved the problem got 25 for a response. The student realized that 25 was too large if one person working alone could do it in 20 days and then got the average of those two. There is a difference between the way students in Japan operate and students in the U.S. operate. Our students look for reasonableness of an answer. I would also like to make a general comment. This notion of looking at the mathematical model is an important one. This whole notion of modelling is growing in mathematics education. Your approach to modelling is a little bit different from some of the approaches that we have taken and your idea of looking at how students deal with assumptions and relating the idea in a mathematical solution to the real world situation is a very important one. I'm very happy to see that you're bringing this to our attention in mathematics education.

Rachlin: I would like to react to what you were saying in terms of the reasonableness. In talking with the students in our classes one of the comments that's happened is that we will find students making wrong answers and go over to them and ask them about the wrong answers. Their comment back is they were happy to have an answer and they were afraid if they checked, they would find it was wrong and not know what to do next, so they don't want to check. That might be a problem of how important just getting an answer is viewed, not the quality of the answer.

Shimada: The same tendency that students do not want to check their work can be found in many Japanese classrooms, though there may be some differences in degree between the two countries. We are also taking it as one of the problems in teaching we need to address.

It seems to me that the subjects whom Professor Miwa studied were better than average.

Travis: I think there is another dimension to this area of mathematical model making also because sometimes our students will get the right mathematical answer but it really doesn't fit the reality of the situation. A very simple example would be something like going to a grocery store and investigating the problem of a "better buy." You can buy a large box of cereal for a certain amount or a small box for much less. Well, a single person can do the mathematics and find that, yes, they need to buy this large box for the "better buy." But it really doesn't fit their real world situation. So I think we have to be sensitive also to having students interpret problems where the mathematics of the idea is one thing, but there is another dimension to it as well. We need to let them be sensitive to that.

Senuma: In the example raised by Professor Travis regarding purchasing either the larger box or smaller box of a product, the key point is to think in terms of the differences in ratio or the differences in values and what's needed. Another aspect of this is that through mathematical means or mathematical solutions one should have the background in this mathematical solution to determine what method to use and in what situations, etc. So developing this type of analytical background in the student is very crucial.

Rachlin: One of the things that I had noticed in the presentation and in the items that you used was the nature in which you are taking problems which may appear in standard texts and then looking at the reasonableness of those questions and moving from those into a model of mathematics. I found that interesting because I've tended to take those questions as questions that were in the text and not think about taking them beyond that to even ask the students about their reasonableness and how they would apply, but I will be doing that now.

Shimada: I would like to comment on the point raised by Professor Travis and Mrs. Senuma, which is the discrepancy between a real and a mathematical solution: recognition of this discrepancy is just the starting point of the process which is referred to as a modification of assumption. The solution found in the usual textbooks may be regarded as the first approximation to the real problem, and afterwards the discrepancy may be noticed. In that case, we will turn to the assumptions underlying the original solution and add some more assumptions or modify some of them to mathematically get the second approximation which we hope will be closer to the reality than the first one. This process will be repeated until we can find a reasonable result.

This may be thought of as a real way of problem solving in a global sense, and we do a portion of it in our classroom teaching.

So it is interesting to start with usual textbook-type word problems, to detect what is assumed in them after solving them in usual ways, and to discuss whether this is a real solution or not. Perhaps some students may say no. In that case, the assumption may be found to be too restrictive or too strong. Then we will weaken it, and try to find what changes will result from this.

If we want to adopt this procedure in the classroom, one feasible way may be to let students discuss the points made by students themselves when they responded to such questions as Professor Miwa asked subjects in his study.

Rachlin: One of the distinctions that I was looking at was how, in this case, you are going from the problems that are in the text, where the mathematical model making that I'm more familiar with went from real life situations and then tried to create text-type problems. This is a big shift, one that would be much more easy to work into a curriculum as it stands now.

Becker: Discussion of the problem, as Professor Shimada suggests, is very important. At the University of Nottingham in England, mathematics educators have developed some very interesting materials on mathematical modeling. One example of those problems is included in my paper, I've forgotten which number it is, but it deals with making a graphical model of a car going around a track. It's interesting, when students try to make a graphical model of the situation, how they have to talk about the different assumptions they are making. Different students make different assumptions and they begin to discuss these. Professor Miwa suggests that we place more emphasis on this with pupils at the upper grade levels of the elementary school and I think the Nottingham materials might have some very good ideas for us. Professor Miwa mentioned in his paper that mathematical problems exist only in the mathematical world and not in the real world, but at a certain point he says, about the ninth grade level, pupils become more aware of the significance and meaning of the model. I'm wondering if he could comment a little bit on why there seems to be a transition at that point.

Miwa: Please look at pages 412 and 413. Some responses of students are shown on these pages. Of course, not all students are aware of the assumptions and significance of mathematical models. But some students began to feel or to appreciate mathematical models. For example, the response of a student of grade nine was that these problems--work problems or walk problems--are good for estimating things. The student realized that the answer to the problem itself is of no use on some occasions, but even though the answer may change in some degree in reality, it provides a good estimation.

Shimada: Whether this change is due to the natural growth of intelligence in students or not, as they get older, is an interesting problem. Since this study is not longitudinal and of a small sample size, the fact that this occurred in the ninth grade may be accidental. It would be an important theme of experimental study to determine at what level of mental growth this recognition can be developed with or without intervention.

Nohda: I think that we might also investigate the nature of the students who make this kind of change. In my experience I find that some students don't like textbook problems. Some of the slower students, actually, are more excited about this matter of model making, that is, that the answer you get is not exact, but only, an approximation. The so-called slower students sometimes are better, in my experience, at this sort of thing.

Rachlin: Any other questions or comments?

Silver: A graduate student of mine was teaching a class of students in algebra, teaching them to solve a problem similar to the work problem just cited and they engaged in a discussion similar to the one that's being talked about here, emphasizing the assumptions of the problem. A very common response among the students in her class was to mention that the cooperation that was implied in the problem between the two people was dependent upon the nature of work that was involved. For some tasks, for example, it might actually take longer for two people to complete them together because of the difficulty in partitioning the work or the likelihood that people would be interacting with each other and therefore taking longer to complete the task than if they worked individually and independently and so on. I wonder if any of the students in Japan mentioned this as an assumption or mentioned any relationship between the answer and the nature of the work that was being done?

Miwa: I feel that similar responses would come from Japanese students. As you can see in some of the student responses given in my paper, they do indeed come up with questions or comments such as "I can't solve a particular word problem without having some specific details about the nature of the work." Or, although they know how to solve these problems, they are quite aware of the need for some detail in tying it to a real situation. And, although many Japanese students are busy preparing for exams, through their preparation for exams they also have picked up the skills to be able to solve these problems, but they still come up with such questions.

Shimada: I would like to introduce a set of problems that I use as an introductory lesson for my model-making class. The first problem is mathematically the same as Miwa's work problem, but deals with a water tank with two spouts with different rates of pouring. The second one is the work problem itself. The third one deals with a walking situation. When two persons are on a friendly walk together, a little different from Miwa's example, how long will it take them to make one round of the park, given the time needed for walking alone in the same route with different numerical values? The fourth one deals with steam locomotives A and B. The times needed for driving a full distance is given to each in different numerical values, and the requirement is to find the time for driving the full same distance when the two locomotives are linked together. Each of the four problems are stated in the same phrasing, as much as possible, and used the same numbers.

After individual study of this set of problems, the students comment on how the problems are similar, how they differ, and why. This serves as an introduction to my model-making lesson and seems a good way to start teaching the role of assumptions in a meaningful way.

Travis: First of all, I do appreciate all of the discussion in this area because I have not considered problems of this type to be mathematical modelling problems, so I have enjoyed this. It tells me that we, first of all, have to stress to students that their answer depends on certain assumptions that have been made in the problem and it gives me a second, but related, thought that mathematics itself does not provide the answer. That means we really need to investigate, at least from my perspective, the ingredients of the reasonableness of answers. These are thoughts that I have never considered before so I do appreciate the discussion.

Rachlin: Any other comments or questions?

Sawada: I have a question to propose to Professor Miwa. As Dr. Shimada stated earlier, that the data used here are taken from a very small sample and that there might be bias owing to such use of small numbers of subjects; do you plan to carry this study further?

Miwa: My answer is in two points. First, please look at page 416. I will read some relevant parts. "Finally, we would like to point out some of the problems not dealt with by this study:

a. Survey of the awareness of assumptions in problem solving needs to be extended to average pupils in public schools. And, if necessary, the provisional conclusion should be revised.

b. The validity of the proposals for teaching problem solving needs to be examined."

Secondly, I think my work will need to be extended to other aspects of mathematical model making, not only to the awareness of assumptions. For example, one is to use a mathematical model in problem solving itself; i.e., a mathematical model as a tool of mathematical problem solving. The other is what assumption, of which students are aware, leads to what models, and what kind of assumption leads to what kind of solution or strategy?

Rachlin: On behalf of the group, let me thank Professor Miwa for his presentation and thank the group for their questions.

Professor Kantowski's Paper

Sugiyama: I should like to begin the session. I am happy to serve as Presider of this session. In this session Dr. Mary Grace Kantowski will speak about mathematical model making in problem solving. I would like to ask you for your kind cooperation. First of all, I am honored to introduce to you Dr. Kantowski as the last speaker. Now she is a Professor of Mathematics Education at the University of Florida. She was born in New York and she was born in the same year I was born. She looks younger than I. Now, I have some sentiment, because in our childhood, World War II occurred, when we, unhappily, stood on opposite sides of the Pacific Ocean. But now in this session, we are at the middle of the ocean. She has secondary teaching experience and went to the University of Florida after being an instructor at the University of Georgia. She is known in Japan through her NCTM work and many excellent papers and books. For example, "Processes Involved in Mathematical Problem Solving" in the JRME. Now, Dr. Kantowski, your presentation please.

MATHEMATICAL MODEL MAKING IN PROBLEM SOLVING

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Introduction

A model is a concrete, visual or symbolic representation of an event, situation, or an idea designed to behave in the same way as what is being modelled. In mathematics, model making is the act of abstracting key elements from a situation and constructing a precisely defined mathematical representation of the situation. A mathematical model may be concrete, such as a solid object made to represent a pattern of numbers; visual, such as a diagram or graph to clarify data or to help visualize something that cannot be directly observed; or symbolic such as a formula representing a mathematical description of a physical occurrence. Modelling is seen in a variety of ways in mathematics instruction. By far the most common application of models in the mathematics classroom is to illustrate a mathematical concept or idea. Other activities include verifying or testing existing mathematical models and constructing mathematical models to represent a given event.

Although most mathematics educators would agree that mathematical modelling is important in problem solving and that modelling should be addressed in the mathematics curriculum, there has been little research into the role of mathematical modelling in the learning of mathematics in general and in problem solving in particular, and even less attention to techniques of instruction in mathematical modelling. The exceptions are the emphasis on manipulatives in the learning of elementary mathematical concepts and operations, and some model making in the learning of geometric concepts.

In the following sections three examples of using, constructing and verifying models in problem solving will be discussed. We will look at (1) model making and problems in number theory, (2) using the computer to model events related to problems dealing with probability, and (3) using mathematical models in the solution of real problems.

Geometric Models in Number Theory

Number theoretic problems in which a solution is often found by finding a pattern and generalizing lend themselves nicely to constructing symbolic as well as geometric (concrete or visual) models. Using concrete or visual models to illustrate the solution to a problem can help students to understand the problem at hand as well as to suggest approaches to solutions to similar problems. For example, the problem of finding the generalization for the sum of the first n integers has applications at several levels and may be modelled geometrically. Two such geometric models are shown below as illustrated by Larsen (1985) in his comprehensive look at the variations of this problem.

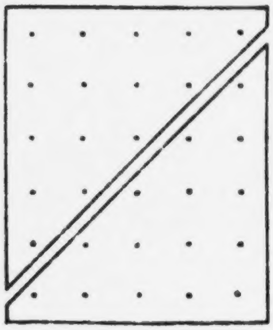

Figure 1

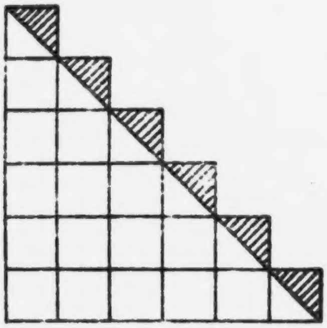

Figure 2

It is clear from Figure 1 that $2\,(1+2+...+n)=n(n+1)$. Therefore, $1+2+...+n$, or the sum of the first n integers, is equal to $1/2\, n(n+1)$. Similarly, Figure 2 shows that $1+2+...+n=1/2n^2+1/2n$. Problems similar in structure to the problem of finding the sum of the first n integers problem are an effective way to reinforce the generalization and to introduce the notions of mathematical structure of a problem and related problems. Many textbooks for middle school and junior high school students (grades 5-9) include a variety of problems such as the following related to this generalization:

1. If there were 20 people at a gathering and each shook hands with all the others, how many handshakes were there?
2. How many diagonals are there in a convex polygon of 50 sides?
3. Given 10 lines intersecting in the plane so that no two are parallel and no three concurrent. How many triangles are formed?

Models for these problems can be associated with the sum of the first n integers to help students to understand the meaning of abstracting the mathematical structure from the context of related problems and begin to get the notion of relatedness. Problems related to the sum of the first n integers problem are excellent vehicles for teaching this notion because of the many examples of problems of similar structure.

Many number theoretic generalizations can be expressed using visual models. Two such generalizations are illustrated below in a feature in Mathematics Magazine called Proof Without Words (Landauer, 1985a, 1985b). Figure 3 below is a visual model of the square of an odd positive integer, and Figure 4 is a visual model of the square of an even positive integer.

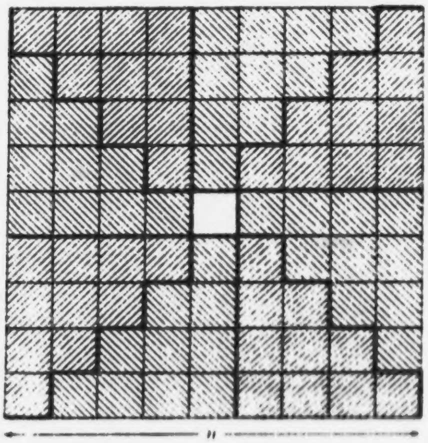

Figure 3

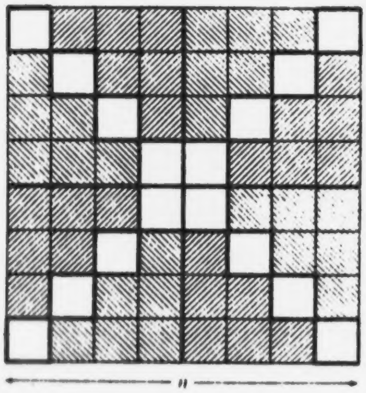

Figure 4

Presenting students with models of such mathematical ideas helps students to think about their meaning and can help students to develop a more visual sense of mathematics.

Depending on the level of the students, they can actually construct such models or be given pieces to be fitted into an n x n square. Students can learn much about the patterns and regularities in the number system by observing and working with the visual patterns in these models.

This application of mathematical modelling can be used to not only illustrate theorems in number theory, it can also suggest directions for proofs of theorems. For example, two geometric models of the proof of the theorem $a+b/2 \geq \sqrt{ab}$ are given below (Figure 5, Eddy, 1985; Figure 6, Schattschneider, 1985).

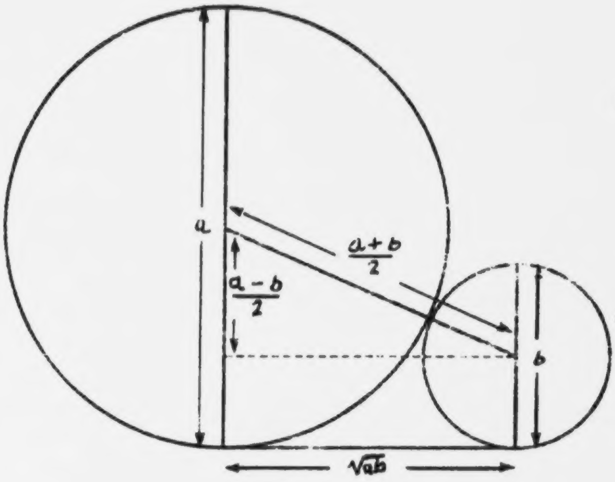

Figure 5

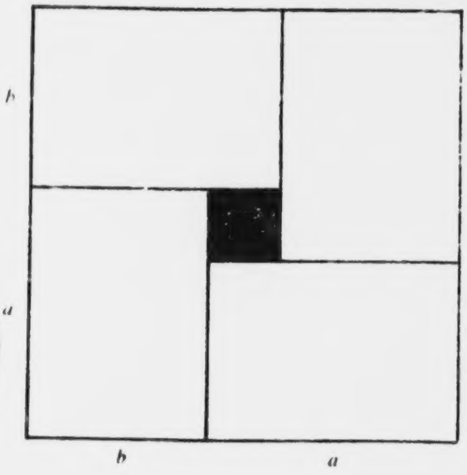

Figure 6

These models not only illustrate the theorem and suggest directions for proofs, they can also help to develop in the student the practice of thinking geometrically and of looking for geometric interpretations of number theoretic and algebraic concepts. One possible instructional technique might be to give students the models and have them explain how the model illustrates the proof of the theorem.

Modelling and Problems Dealing with Probability

The advent of the affordable microcomputer has opened the door to a wealth of powerful modelling possibilities. For example, the randomization capability of the computer allows us to simulate large numbers of trials of events to help students solve problems in probability. The computer is an especially good tool to use with problems for which the solution is counterintuitive. For example, given the following problem:

Problem: Suppose I have two six-sided cubes. Each cube has a square (■) on three of its faces, a triangle (▼) on two if its faces and a circle (●) on one face. If the cubes are rolled 1000 times, what combination of shapes will occur most often?

Students generally guess that the ■■ combination will occur most often and are surprised to see that the ■▼ combination is dominant. Using the computer to model the rolls of the dice in this problem awakens the students' curiosity about the mathematical principle behind the correct solution.

Another classical problem that can be modelled using the computer is the "prize in the cereal box" problem

Problem: For a limited time the Cherry Berry cereal company is including a commemorative button in each box of Cherry Berry cereal. If there are 5 different commemorative buttons, what is the average number of boxes of cereal I would have to purchase to be assured of getting all 5 buttons?

Many students believe that they need purchase only 5 boxes of cereal to complete the collection. Moreover, in observing the computer simulation, they are often surprised at how the number of needed boxes changes in different trials.

Problems such as these are excellent for students at the middle school and junior high school level because they give them some background in elementary concepts of probability and prepare them to approach the subject more rigorously later.

Constructing and Verifying Models

The problems illustrated above are of the type that Burkhardt (1979) calls *dubious* (problems that provide exercise in mathematical technique), or at best *educational* problems (dubious problems that make an important point of some principle). He sees dealing with "real" problems (those related to everyday life) as providing material as the most important type of modelling in problem solving. The models were used to *illustrate* a mathematical idea. Problems dealing with real world situations most often require students to *verify* existing mathematical models of an event or to *construct* a mathematical model to reflect a situation.

In the *Challenge of the Unknown* (1985), a film series dealing with problem solving in real life situations, several of the film segments deal with excellent applications of model making in mathematical problem solving by illustrating how a physical event or situation can be represented by a mathematical model.

In one segment, a paleontologist is testing his theory that dinosaurs were actually very swift animals and not the slow creatures we imagine them to have been. Since dinosaurs can no longer be observed directly, the paleontologist uses an ostrich as a concrete model of a running dinosaur. The Alexander formula provides a mathematical model of the velocity of a running animal if the animal's hip height and stride length are known. Alexander found that the velocity of any running animal may be approximated by the formula:

$$v = .25\ g^{0.5} s^{1.67} h^{-1.17} \text{meters/second}$$

where g is the acceleration due to gravity, s is the stride length, and h is the hip height of the animal. This model may be used in several ways in the classroom. First, it may be verified by having students take measurements of their hip heights and stride lengths, calculate the ratio, and graph the results against predicted speed to form a curve of best fit. Students can also calculate velocities of running animals by using Alexander's formula. Such activities give students experience with the concept of how mathematics can model a physical event in two different ways--visually, by means of a graph of what is happening, and symbolically by using the equation to represent the velocity. Moreover, they help students to understand that a mathematical model is an estimate, that the model is an ideal and that there is some discrepancy between the model and reality.

Another Challenge of the Unknown segment deals with a real problem in sports--that of increasing the speed of a downhill racer in skiing. A technical engineer uses a computer to create a model of a speed skier on a course. He uses baseline data of a championship skier to find out what he can about other conditions and other skiers. He uses graphs to show how variables such as air temperature, body weight and drag affect the speed of a skier. Such models can help skiers to predict what will happen under various conditions.

Studying approaches such as these to real problems help students to see the usefulness of graphs and mathematical formulas to model real life situations. In the typical first year algebra class, motion problems are generally introduced as a type of problem that can be solved using two equations in two variables. At another point in the course the Cartesian co-ordinate system is introduced, and students are given practice in the graphing of linear functions. Both are important mathematical ideas, but they are seldom tied together in the algebra curriculum and students generally see them as abstract operations with little application to solving problems. Introducing graphing as a way to model problems dealing with physical phenomena such as motion and to predict outcomes could be a way to motivate students by making the study of algebra more meaningful.

The Role of Mathematical Modelling in School Mathematics

The role of mathematical modelling in school mathematics today is in roughly the same position as problem solving was in the school curriculum twenty years ago. Since that time there has been significant research into the processes involved in mathematical problem solving (e.g., Clement, 1982; Kantowski, 1980; Schoenfeld, 1979). Much has been learned about the processes used by successful and unsuccessful problem solvers, about the thinking of so called experts and novices, and about the problem solving ability of students at various levels of mathematical aptitude. Researchers have found that successful problem solvers employ a variety of general and specific heuristics in solving routine as well as nonroutine problems. These heuristics are rules of thumb that are useful in the solution of problems. Researchers have also found that experienced problem solvers and those of high ability exhibit processes that are essentially different from those employed by less experienced or less able students.

In the study of problem solving processes, and in the application of what has been learned to classroom practice, researchers and curriculum developers generally assume a model of problem solving based on or similar to a model proposed by Polya in his classic work How to Solve It (1957). Polya observes that successful problem solving generally occurs in four phases during which the problem solver interacts with the problem in some way. He suggests that the problem solver ask himself questions related to the information in the problem that will help in the movement towards a solution. Polya's four phases and some suggested heuristic questions are included in Figure 7.

POLYA'S FOUR PHASES IN THE SOLUTION OF A PROBLEM

Understanding the Problem

- Do I have a mental picture of what is given?
- Would a figure or a diagram help?
- Do I understand the conditions?
- How is what I am given related to what I am looking for?
- What is the unknown?
- What are the conditions?

Devising a Plan

- Have I seen a similar problem?
  - How did I approach the solution?
- Can I solve a part of the problem?
- Have I used all the data?
- Have I checked the conditions?
- Have I taken into account all the notions implied by the data?
  - All relationships?
  - All associations?
- Could I organize the data into a table?
  - Find a pattern?
  - Find a generalization?
- Can I think of other data appropriate to the unknown?
  - E.g., If I had...

Carrying Out the Plan

- Can I follow through each step of the plan with the correct computation?
- Is each step correct?
- Can I prove that it is correct?

Looking Back

- Can I check the result? The argument?
- Is there another way to solve the problem?
- Is there another, distinct solution to the problem?
- Can I think of a different, related problem?
- Could I use the result to solve a problem that I have seen before?

Figure 7

Polya's suggested questions in the first, second and fourth phases of problem solving are heavily dependent on mathematical model making. In the understanding phase the problem solver is asked if he or she has a mental picture of what is given and if a figure or a diagram would help to produce such a mental picture. Thus, the problem solver is asked to construct a model of the problem situation. In the planning phase, the problem solver is asked to recall a similar problem, that is, a problem of similar mathematical structure. In order to relate a problem to others of similar structure, the problem solver must be able to abstract the underlying mathematical model from the context of the problem. In the fourth, or looking back phase, the problem solver is asked to relate the mathematical model of the given problem to other situations and to other problems that have been encountered before.

Over the last two decades a steady stream of research on the relationship of the use of heuristics to successful problem solving and on the effect of explicit instruction in the use of heuristics has resulted in an ever growing body of knowledge about the processes that are used during problem solving and about how to teach problem solving effectively. The introduction of several curricula with a focus on problem solving (e.g., Lane County Mathematical Problem Solving Project, 1984) and a greater emphasis on problem solving in textbooks and at professional meetings provide evidence of widespread implementation of what has been found. One form of evidence that this implementation is successful comes from the fact that variations of problems that were previously attempted only at higher levels are now being presented to middle school, junior high school and high school students who are able to share in the excitement of experiences in mathematics previously denied to them.

It is clear that mathematical modelling is closely allied to problem solving. However, how they are allied and the effect that explicit instruction in modelling might have on a student's ability to solve problems is not known. These and many other questions must be answered about mathematical modelling before issues of curriculum development can be addressed. A concerted research effort such as that which was undertaken in problem solving years ago is needed.

Needed Research

As both Burkhardt (1979) and Treilibs (1979) have noted, there has been very little research done on the role of mathematical modelling in problem solving. In looking at the state of mathematical modelling and its relationship to problem solving, several interesting questions present themselves:

(1) Krutetskii (1976) notes that students of high mathematical ability have a "mathematical turn of mind," or the ability to mathematize a given situation. Moreover, Fruedenthal (1968) sees the mathematizing of reality as a very important part of the mathematical experience. Years ago, we asked if heuristics could be taught effectively. Today, an analogous question presents itself with respect to the teaching of modelling. Can students be taught to mathematize a situation, that is, to create a mathematical model given a situation? If so, does a student's problem solving ability improve as a result?

(2) Great strides have been made in helping students to become better problem solvers by explicitly teaching general and specific heuristics as part of problem solving instruction. General problem solving heuristics such as those proposed by Polya as well as specific strategies for problems of a given mathematical structure have been included in instruction. For example, the guess and check and find a pattern and generalize strategies have been successfully taught in many curricula that emphasize problem solving (e.g., Lane County materials). What are

analogous instructional techniques to teach students to create mathematical models of given problem situations?

(3) Given the availability of the microcomputer in the middle school and secondary classrooms, studies are needed to look at how students learn mathematical concepts that can be taught by modelling using the new technology. For example, many problems involving number theory and probability can be modelled, and the verification of models requiring complex mathematical formulae are now possible using the computer. Do students using the computer to model a problem situation solve problems differently from students who approach such problems purely theoretically?

(4) Pollak (1968) sees giving students prepared models as depriving them of the essential experience of learning to create their own models. Would having students define mathematical models for "standard" application problems rather than giving them such models make students better problem solvers? A teaching experiment in which students are asked to develop predictive models for routine application problems could provide interesting data in this area.

(5) Studies are needed to find out if students will become better problem solvers if they are overtly taught to look for the mathematical structure of a problem. Studies that look at what problems students perceive as similar have been done (e.g., Silver, 1981). What is now needed are studies to determine effective instructional techniques for teaching students to recognize similar structure. One hypothesis is that model making could be one such effective technique.

Summary and Conclusions

Activities in which mathematical models are applied in some way are clearly important in problem solving. Interesting and creative solutions to problems are often those that begin by formulating a new and interesting model of the problem situation. What is not clear is how model making fits into the scheme of what we know about problem solving processes and how model making should be taught. There is a need to organize what we know about the role of model making in learning mathematical concepts and to formulate questions about what we need to know about making and using models and the teaching of mathematical ideas as well as the relationship of model making to problem solving.

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Discussion of Professor Kantowski's Paper

Sugiyama: Thank you very much for that very enlightening presentation and, I must say, you speak English very well (laughter). Now for the discussion.

Hashimoto: If the great Buddha in Nara could walk, then how many hours does it take for Buddha to walk from Nara to Tokyo? If the sitting height of the Buddha is approximately 16 meters, then the height of the Buddha will be 2 times 16 = 32 meters. If the height of a man is 1.6 meters, then the height of the Buddha becomes 20 times as tall as the man, because 32 divided by 1.6 equals 20. If the man walks about 5 kilometers per hour and the speed is proportional, then the speed of the Buddha is 20 times 5 = 100 kilometers. As the distance between Nara and Tokyo is about 400 kilometers, the Buddha takes 4 hours on foot, because 400 divided by 100 equals 4. On the other hand, it takes about 4 hours by the National Railroad. Then the time of the Buddah is the same as that of the National Railroad. This is an advertisement of the Japan National Railroad which I saw this spring. In this story there are some important assumptions and this is one example of mathematical model making.

Wilson: In the materials that Dr. Becker talked about from IOWO yesterday, involving the giant, many of the same kinds of modelling processes are used.

Rachlin: I had an opportunity to use the one computer program that Dr. Kantowski loaned me of the cereals problems with the pens or the buttons in the cereal box. I used it over on the Big Island (Hawaii) with a class of eighth grade students. The model looked a little nicer as we went to do it. What had happened with that is after the students finished the problem and we had worked on it for a while (I was only there the one day), later the teacher told me that the students had decided to go out and get some cereal boxes and test this out. They decided that the model wasn't working and tried to figure out why it wasn't working. They felt there were some other variables that were also in that problem besides what's there. Have you had similar results or reactions?

Kantowski: Yes, in fact, we did have similar results. Many times what happens is that one color is not available. Students will buy many, many boxes and there will be one color that is not available. Now, there can be many reasons for this as you see, a company could be deliberately keeping that color off the market so you will buy more cereal, or it just happens that when cereal boxes are sent to various places that one color does not get sent to an individual store. But what is interesting about a computer simulation is that the program for the simulation is very easy. It will all fit on one screen with very few lines, and the students can discuss what can be changed in the program to make the simulation much more real and much more like what is actually happening in their situation. So that instead of giving every one of the colors an equal probability of occurring, you can get a small probability occurring for certain colors. They can decide on what probability to give each color.

Hashimoto: Regarding this matter, I think that the color problem is related to random numbers and in Japan we don't teach this content. Do you treat Monte Carlo methods in school mathematics?

Kantowski: That depends on the level of sophistication that you are interested in. With very small children we will do things like tossing a coin and finding out how many heads and how many tails occur and also other kinds of very simple probability experiments. When we get into the junior high school, we begin to do problems like this. These would be problems that would be taught at sixth, seventh or eighth grade level. I have used these problems with teachers at that level for use in their classes. The more theoretical aspects of the Monte Carlo method are not looked at until secondary school and maybe not even then.

Yoshikawa: I have one question. You have mentioned the use of computers in teaching mathematical problem solving. I wonder whether students should make programs by themselves or teachers should give the appropriate software to students. What is your opinion?

Kantowski: I think that the computer should be used in both ways. There are times when there are very good programs that are already prepared and the data generated from those programs or ideas that can come from using those programs can help students learn mathematical ideas so that prepared programs are very good. Also there are some problems that students can solve by writing a simple program. I don't believe it is necessary for students to do complex programming to be able to use the computer effectively to solve mathematical problems. Simple programming in BASIC or LOGO or Pascal (those are the three languages that we use) can help solve very difficult problems. Thus, students can use programming and the computer as a tool. I don't believe that programming should necessarily be the goal, but using a simple program to solve a problem should be the goal. Programming can be used at every level. Even very young children can do LOGO programming and see and learn some mathematics from programming in LOGO.

Miwa: Thank you very much, Professor Kantowski, for your interesting presentation. I am much impressed. I have two questions, which may be regarded as comments. The first is on using microcomputers in model making. During these last two years I carried out development of mathematical model making materials supported partially by a Science Grant. But I encountered several problems or difficulties. One is how to use microcomputers. In model making, situations are often represented by difference equations or differential equations. In some cases, equations are of a non-linear form. In that case, of course, school students cannot solve the equations. And solutions would be discovered by using microcomputers. I hope you will kindly show your examples of this type.

The second is related to your visual model. As you pointed out one kind of model is a visual model. I think that some situations would be expressed in geometric figures or, using my terminology, geometrization of situations. For example, when you drive a car and turn to the right, the front wheels of your car and the back wheels do not move on the same loci, and the back wheels run more inside than the front ones. So, if a man stands on the right side near the car, then the back wheels become dangerous. This situation will be expressed and comprehended within geometry, not using other physical factors such as force, speed, etc. I call such model making as this geometrization. What are your reactions about this geometrization? If there are criteria, which is the most important?

Kantowski: Are you asking what is the most important idea about presenting a geometric interpretation of a situation?

Miwa: Yes.

Kantowski: First of all, students need to learn at a very early age that it is possible to present something visually. If students at the secondary school have not had experience in seeing a visual representation of a physical event such as a speeding car, it would be difficult for them to interpret that kind of a visual model. So I think that it is very important to start young. You have brought up the important point that modeling is good for situations in which it's dangerous to try to actually do something. That is one way that the computer is used in industry. We can use such predictive models with students and, as you mentioned, the model doesn't have to be a simple one. Previously if we were trying to graph a situation or model it, the graph would have to be very simple; but the computer can handle any kind of data and can be used as a predictive model with any kind of data. To answer your question simply, we can collect data in situations where it is possible to measure and then use the model as a predictor in situations where it is impossible to measure. That's one way that we can use geometric data. Also, in the situation of the speed skier, for example, as you can see from the sheet that I gave you, there is something about a visual model that makes what is happening much clearer than when you see a set of numbers.

Silver: I'd like to mention two projects in the U.S. which are developing materials very similar to two of the themes that Dr. Kantowski referred to in her paper. One is at the Educational Technology Center at Harvard. They have been developing computer software for use with students at the middle grades and at the high school level which involved the simultaneous representation of mathematical ideas in numerical or algebraic symbolic form and in graphical or pictorial form. Using a split screen effect, a child can go in and choose to manipulate either in the graphical form or in the symbolic form and see what the effects will be on the other representation of that manipulation either by adding or transforming terms. I think that's a powerful way in which the computer technology can be used to accomplish one of the goals that was mentioned in the paper. The second project is one that is also housed in Massachusetts which is being done by the Consortium for Mathematics and its Applications which has previously developed a fairly large number of independent study modules on applications of mathematics for use in the undergraduate level in the U.S. It is now working on another project specifically aimed at high school applications. Those materials are of the sort depicted in the video that we saw. But they are not videos, they are in fact instructional packages that can be used by teachers with students at the secondary school level. Both of those projects are, I think, going to produce materials that are going to be useful along with some of the other ones like we have heard about, such as the materials from Nottingham and IOWO and many of the things that Dr. Kantowski has already produced in her work.

I have a question that I'd like to ask because I am confused. In your paper you mentioned Krutetskii and the mathematical *"cast of mind"* or *tendency to mathematize*. That has always appealed to me as being about right but, at the same time, I find I'm confused. In Professor Miwa's data I get the sense that there are many students who are quite mathematically capable, talented students, who do not see the relationship between the mathematics that they are studying and the real world, and I'm confused about how these two different perspectives can reside simultaneously and both seem about right. Can you help?

Kantowski: I don't really know, but it would seem to me that there may be some people who are born with the ability to mathematize or who can develop the ability to mathematize very early, and there may also be some children who have a latent capability to be able to do that but they need instruction to be able to actually perform the mathematics. I am not sure how much instruction Professor Miwa's students have had in the actual mathematizing of a real situation so some instruction may be necessary for people to be able to mathematize. I don't know the answer to that but I think it's something that I would like to look into. It seems to me that the ability to mathematize can be developed.

I would like to make a comment on your earlier description of the materials being developed at the Harvard Center. I think that this use of the computer is probably one of the most powerful that we have to help teach mathematics. You talked about taking data, manipulating it and looking at the results of that manipulation. This relates to what Professor Shimada said in that one of the ways that we can change our assumptions and move closer in approximation to a predictive model is to use the computer to do this. We can use the computer to manipulate the data with our changed assumptions and see what happens with the model. The computer allows us to do this very quickly and in a way that we can study the results of manipulation.

Becker: I think Professor Silver was referring to talented students, students who are identified as talented who have this mathematical turn of mind and talented students who don't seem to exhibit it. There are also talented students who are not yet identified as talented and I wonder whether some of the ideas that we have discussed here in the papers of Professors Kantowski, Miwa, Travis, and Nohda might provide an opportunity for that talent to show itself. Finally, a couple of years ago when Professors Miwa, Shimada, and I and others had some talks about the possibility of a seminar like this, we discussed the role of pattern finding and modeling as perhaps two of the important ingredients in the seminar agenda. I was quite interested in those ideas at that time and the interest has continued to grow. Now I wonder whether this afternoon we might want to try to get a little bit more specific about how we can propose to do some cross-cultural research in these two areas of problem solving.

Sugiyama: Time is up. Thank you for the interesting discussion.

Discussion of Cross-Cultural Research And Follow-Up to This Seminar

Becker: Professor Miwa and I would like to have an open forum for discussion of the proceedings, the organization of the proceedings and whatever comments you would like to make about them.

Kantowski: In the outline of the parts of the proceedings, are they in the order in which you wish to put them in the book? Will there be an introduction? You have an overview of the seminar which includes what went on at the seminar, but will there be an introduction?

Becker: Would you suggest that? I thought Professor Miwa would have an opening statement and I would have an opening statement, each on behalf of the delegations.

Kantowski: Yes, I think an introduction would be very important. The other question I had relates to the distribution of the report. We talked about that a bit, but I would like the Japanese delegation to have input as to how these proceedings are going to be disseminated in each of the two countries.

Becker: The American group has spoken a little bit about dissemination of the report in the U.S. We have reached no conclusion. Professor Miwa, I am wondering if you and your Japanese colleagues might want to indicate how you wish to have the report disseminated in Japan?

Miwa: If the proceedings are going to be published in book form, we foresee no problem in getting it out to all the colleges, universities, teacher training centers and other institutions that pertain to the area of mathematics education.

Becker: Professor Miwa, do you have an idea of how many copies you would need for distribution?

Miwa: At most 200. Will the report be in English? We haven't quite decided among the Japanese delegation whether we should translate the report into Japanese. We have no opinions about that. So as far as number, going back to the original question, we were talking about a report in English of, say, 200 copies.

Whitman: If the Japanese papers have, I assume, been written in Japanese, then there is no translation involved. Could your publication of the proceedings have at least the Japanese part both in Japanese and in English? I think that would make it available for more people in Japan in a shorter period of time than waiting for the complete translation of the Japanese, but maybe I should ask the Japanese to speak on that.

Shimada: We Japanese participants have not yet discussed the publication of a Japanese version of the proceedings, but we must think of it as a possibility. If we try to publish the Japanese version, all papers, including the Japanese ones, must be translated because the Japanese papers presented here were not necessarily first written in Japanese. What I am worrying about, in this respect, is the matter of copyright. Perhaps this would need the consent of all concerned.

Becker: American reactions, suggestions? We have not made a final decision on where the report will be printed, under whose auspices, and so on. Originally, in the proposal that went to the National Science Foundation and to the Japan Society for the Promotion of Science, I think it called for funding to print 75 copies of the proceedings. In a couple of cases, the reviewers of the proposal suggested that there might be more widespread interest in the proceedings of the seminar and maybe we should think of a larger number.

Originally I thought that it could be printed at my university and disseminated from there. A suggestion has been made that perhaps we could contact the National Council of Teachers of Mathematics to see whether it might be interested in publishing the proceedings. Another possibility, of course, would be to contact a commercial publisher. What are your reactions to these possibilities?

Miwa: At most one hundred or so copies would be sufficient for the Japanese. Books written in English are not so widely read in our country. But the content of our proceedings is interesting, and hopefully a considerable number of people would like to read and study them. As to the translation into Japanese, in Professor Shimada's opinion, its possibility is not now known, because it is very hard work and would cost much money to translate not only papers but also the discussions into Japanese. We must find some source of support. So to publish a Japanese version of our proceedings cannot be determined now.

Becker: We still haven't addressed the question of copyrights. I am not sure how to respond to that. Professor Shimada, do you have some suggestions?

Shimada: If we agree with publishing a Japanese version and have consent on the copyright, we will try to translate all papers into Japanese and seek a publisher or an agency to help us. If this is not possible, then we will try a less expensive style of publication. Anyway, we would like to distribute the outcomes of this Seminar to all Japanese who have interest.

The next copyright problem may occur before the publication of full papers. Many of us will be requested to write an article or make a lecture about an interim report of this Seminar by various sources in Japan.

In that case, parts of papers may be cited in that article or lecture. Perhaps partial citation will provide no problem. But in the articles or in publishing lecture notes, as is often the case for supporting agency of lecture meeting, a lengthy citation may occur. In this case, there may be a copyright problem depending upon the amount of citation. So I would like to have concensus on this matter.

Becker: My feeling is that it would be wise for us to keep it as uncomplicated as possible. The Japanese have indicated that perhaps they would like to have 100 copies for dissemination in Japan. When we Americans discussed it a few evenings ago, we thought more than 75 but certainly not 500. Something on the order of 200 would be a reasonable estimate. So perhaps a total of 300 copies could be printed within our budget. I think it would be possible to come up with an appropriate and attractive cover layout but use an inexpensive duplicating process. In the event there is a larger interest than we anticipate, I think more copies could be printed at that time. Let me make a suggestion to get your reaction. Concerning the copyright question, perhaps on the Japanese side you could explore the situation in Japan and we can explore the situation here in the U.S. Then Professor Miwa and I can be in contact to see what seems to be a reasonable thing to do. Let's have reactions.

Wilson: I am not sure I fully understand the copyright laws in the U.S., and I certainly don't understand them in Japan. When I was editor of the JRME journal, each manuscript that came in had to get a copyright release from the author. For individual articles the copyright can be maintained by the author, but we indicate that there. If they give the release, then the whole journal is copyrighted by NCTM. That's how we handled it when I was editor of JRME. Staying slightly away from the copyright question, I would urge that we find channels for getting this out expeditiously. If we go through NCTM or Lawrence Erhlbaum publishers, it adds a year to the time that it gets out. That's what I am concerned about and I would rather see it in a reasonably inexpensive binder and available in a short term. The second point on that would be our colleagues that might want it. They will be willing to put a few dollars out for it, but if it is an expensive publication from NCTM or Lawrence Erhlbaum, then fewer of them would take advantage of it.

Silver: I would like to make a suggestion that we probably won't be successful in resolving the copyright queston, so I agree that the thing to do is to pursue them separately. But I want to raise a couple of points and then maybe we can move from this on to something else. There are two points that haven't yet been raised as far as copyright is concerned. One has to do with the publication of the papers in other forms, some other publication, and in addition to the problem that Professor Shimada spoke about in terms of quoting liberally from papers, it may be that some papers would be published in some other outlet, in a journal or as a chapter in a book or in some other form, and that presents another copyright problem. The second point is that there is a difference, an important difference between material that appears originally in our papers and material which is, in fact, copyrighted already. For example, in Dr. Clarkson's paper, in her Appendix she includes many examples of already copyrighted material. In other cases, we have had presentations such as the one tody by Dr. Kantowski in which she used copyrighted materials as examples during the presentation and which, in fact, were discussed in the discussion which were not part of her paper. So there are these other issues that are going to complicate both the copyright question in terms of what can and cannot appear in the papers and also the question of the conformance of the discussion to the papers because there have been those things that have happened here which are not part of the papers, so that's one more.

Clarkson: I understood when I prepared my paper that part of what I was sharing here might not be able to go into a final report. I would certainly be happy to change my paper to conform to whatever is needed. I feel very strongly that it is important that a report reflects what actually occurred. I think this has been a very important meeting. I know it has been an important meeting for me. I believe it will be a very interesting combination of papers for anyone interested in Japanese mathematics education or in mathematics education in the U.S. Our first responsibility is to make sure that there is a paper for our proceedings here. Maybe another version of it will need to be prepared for some other publication.

Becker: Let me make a suggestion and you can give me your reactions. With respect to the papers being published somewhere else, let our attitude be to let people do that. If authors would like to submit their paper prepared for this seminar to another outlet (e.g., to a journal) it's their option to do that. It will probably happen that the paper would appear first in our proceedings. Then any such questions about copyright would have to be handled after the Japanese and the Americans have had a chance to explore the question. With respect to materials included in papers already copyrighted, my suggestion would be this - that each one of us is to have his or her paper revised and in final form by October 1, 1986. At that time, if the paper contains a significant amount of such material, we'll cross that bridge at that time. What is your reaction?

Sawada: My personal feeling is that this is not the time or the place to discuss or linger over the copyright matter, and that to give precedence in the past whenever such international conferences have taken place that proceedings have never been translated into Japanese. If doing so is going to cause a certain copyright problem and if it seems like a timely idea to produce such a translated version, let's cross that bridge when we get to it and move on at this point.

Becker: To quickly summarize, the spirit of the mechanics of the proceedings is that we will try to get it out as soon as possible. We will aim at approximately 300 copies, 100 for the Japanese, 200 for the Americans. If an author wishes to have his or her paper submitted for publication elsewhere, fine, and that questions of copyright will be worked out over time. Are there any other questions that we should discuss in regard to the proceedings?

Wilson: Do you have any guidelines on the extent of revision of papers that you would like to see? Are there any guidelines for what this revision process is to be and the same question would apply to the discussions? Is the revision primarily for purposes of accuracy, or can I really change what I said or wish I had said in discussions?

Becker: I would suggest that each author make his or her own decision on the revision so long as the paper is completed and sent in by October 1.

Wilson: My paper was not read, therefore my presentation was somewhat different from the paper itself. Is a copy of the tape available to me?

Becker: We can have a copy of the tape made. The only consideration, of course, is the time element.

Wilson: But am I correct, the presentations are not being transcribed, it's only the discussion that is being transcribed?

Becker: Yes.

Rachlin: I have some trouble understanding the discussion about transcription. I can understand the transcription but it's just that there will be no translation that needs to be done then on anything other than checking translations. Since all the discussions have been translated and all the papers have been translated already, then English versions of everything are available either in written form or on the tape, is that right?

Becker: The papers are in English and they will appear as they are submitted in revised form. The transcriptions will be made and everyone will get a copy in order to check for accuracy and to fill in what spaces need to be filled in and so on.

Rachlin: And that's in English as well?

Becker: Yes, taken right off the tape.

Rachlin: I guess what was working through my mind was still if there were to be a Japanese version, the same thing is true for the Japanese, the discussions don't need to be translated, they've already been done during our discussions. The only new part to be translated will be the papers themselves.

Becker: Yes, I didn't understand what your point was. Now I understand it better. Any other questions? One of the participants inquired about the length of the papers that will appear in the proceedings. The suggestion has been made that we should try to limit the length to twenty pages, double-spaced. What are your reactions?

Hashimoto: My paper as you know includes a transcript of the classroom recording. It has been shrinked to within twenty pages. Is it all right?

Silver: Relative to Professor Hashimoto's comment, if you don't place other restrictions then the number of pages and the double-spacing aren't really meaningful unless there is some other uniformity across all the papers, like the size of the margins and the type size and style and so on. Otherwise I can take my 35-page paper and reduce it or otherwise produce it on 20 pages and that hasn't solved the problem, or has it? Is it really just an issue of the final size of the proceedings or is it that we want to have papers all about the same number of words and pages and so on? What is the issue?

Becker: I think the question was raised so that we could guard against the thing getting too thick, but my own feeling is that the papers are not of such length that that is going to be a big problem.

Rachlin: At this point, there is great variability in the looks of each paper just as you pick it up and start looking at the type on it and depending on what computer it was printed on. Is it anticipated that at some other spot it will be typed into consistent format and then that might answer some of those other questions about spacing and things because you would already be setting the margins when you type them in or have them typed in, and I think it would change page numbers considerably once the type was changed?

Becker: I had anticipated that all the papers would be retyped after I have them. I don't foresee any problems. Does it seem alright? OK, then I have a brief announcement. A couple of people have asked about getting more of the beautiful postcards of the East-West Center. It is now possible to get more but they will cost twenty cents each and we have to let the people in the office know how many we want. So anyone who wants postcards of the East-West Center should let me know right now. Thank you.

Professor Miwa, would you like to introduce the document about the findings and proposals coming from the Seminar?

Miwa: Please look at the paper distributed at the beginning of the afternoon session. This is the Findings and Proposals of the U.S.-Japan Seminar on Mathematics Problem Solving. Of course, it is a preliminary draft. First I will read it and then please give your comments and questions. After our discussion it will be revised.

U.S.-JAPAN SEMINAR ON MATHEMATICS EDUCATION
(Problem Solving)

Findings and Proposals
(Preliminary draft)

1. Findings of U.S.-Japan Seminar:
    a. The importance of problem solving is recognized unanimously by all participants. There may be some delicate differences among participants regarding the following:
        - What is problem solving?
        - What is a strategy?
        - Is problem solving activity a goal itself or an important means to achieve a higher goal.

    b. Language seriously affects children's process of thinking. Care must be taken in this and how to handle it is still open.

    c. The importance of teacher training is recognized unanimously as being important, especially having actual experience at problem solving suitable to each teacher's level. Reflecting on the process of arriving at a solution is indispensable in order to be a sympathetic helper to children who may feel difficulties in problem solving. Special attention is needed for novice teachers.

    d. Selection of problems should depend on the objective(s) of teaching at the time. Various kinds of problems may be useful to challenge the curiosity of students and their inquiring minds if they are well-organized. How to relate problem solving more realistically to students and to make it interesting is an important question on which we want to continue to focus.

    e. In teaching problem solving, we need to use methods to encourage inquiry and to help students to be conscious of the processes which are used for arriving at solutions. In this manner, we may teach a kind of strategy.

    f. Each side wishes to continue to learn more about how thinking and problem solving are approached in the mathematics curriculum and classrooms in each country.

    g. Both sides agree that evaluation of problem solving is important and should be a focus of our interest. For example, how is pattern finding evaluated?

    h. Both sides agree that a need exists to see how classroom teaching is carried out in each country.

    i. The goals of mathematics education need to be analyzed to see how problem solving fits into the framework.

    j. There is a need to consider further the question of teachers' activities in the classroom - children centered or teacher centered?

    k. Both sides agree that interaction between members of the Japanese and United States mathematics education communities is important and that collaborative research holds important potential for expanding our knowledge of learning and teaching problem solving.

1. Both sides agree that a Japan-U.S. Joint Study Group for Mathematical Problem Solving should be formed. The Joint Study Group will provide a facilitating and enhancing vehicle for formalizing and carrying out cross country activities of all kinds in mathematics education. The Joint Study Group will serve to continue the friendly and scholarly collegiality generated in this seminar.

2. Proposals for further action:

   a. A series of problems should be developed, of a non-routine nature, that can be used in the classroom, in teacher training, and in research at the various levels of school and teacher training.

   b. A program of collaborative research into children's problem solving behavior needs to be planned. We should target the following populations·

      - lower primary grades
      - middle primary grades
      - upper primary grades
      - one grade of the junior high school level
      - one grade of the senior high school level
      - students in preservice teacher education programs

   c. An exchange of inservice mathematics teachers and mathematics education researchers should be planned.

   d. An exchange of video-tapes of classroom scenes on problem solving should be planned, to include novice as well as experienced teachers.

   e. An information network should be established through which ideas, information, materials, and progress in cooperative, joint work can be communicated. For this purpose, Professor Miwa, Mr. Sawada (Japan) and Professors Becker and Wilson (USA) may serve as focal points with the responsibility of facilitating information/materials flow.

   f. Both countries have exceptionally good teachers. The question of what makes an exceptional teacher should be addressed within and between countries.

   g. In order to acquire a better understanding of material(s) and teaching in the mathematics curriculum of each country, the question of translation of materials should be addressed in a realistic manner:

      - Consider translation of only the problem solving material in textbooks and teacher training materials from each side.
      - Consider the possibility of translating test materials from each side.

   h. There is a need to pursue funding possibilities to support and carry to reality the outcomes and future plans of this seminar.

   i. We need to take care to:

      - widely disseminate the proceedings of this conference
      - look for opportunities to exchange mathematics educators from the two countries.
      - to explore, as possible and in a timely manner, other societal/cultural characteristics that relate to problem solving behavior; e.g., expectation of teachers and parents.

Discussion concerning Seminar Proceedings:

papers (due to J. Becker by October 1, 1986)

discussions - transcribed and finalized after each participant has had an opportunity to review.

highlights of the Seminar - a short paper (3-5 pages) agreed to by participants.

tentative publication date: July 1987

outline of parts of proceedings

- papers
- discussions
- participants/photo
- overview of Seminar
- Seminar findings
- Seminar follow up activities

Becker: This Preliminary Statement is based on notes that a number of us took during the discussion yesterday. The Japanese then prepared a little working document and following that Jim Wilson and I went over it taking into account our notes of yesterday's discussions, and we made some adjustments. That's the background for this paper.

Nohda: Concerning item 2d, included is the term novice. Maybe the "novice" is not so appropriate, because I think the term "novice" is used differently in Japan from the U.S. I propose to replace the term novice by pre-teacher.

Becker: So let us replace novice by student teacher? Is that acceptable?

Nohda: It would be difficult to get the data of novice teachers, so I am proposing to change the term.

Wilson: I think it's not so much the specific definition or wording here that's needed but the general idea of exchanging samples of teaching problem solving or teaching generally in mathematics at different levels. It would be fruitful to me on, say, one project I think to look at the best kind of teachers in each country and do some comparisons with a colleague who has the language facility to understand and explain what's going on. And it would also be interesting perhaps in a different project or an extension to look at student teachers or beginning teachers.

Shimada: I would like to propose a change of expression in 2d. It would be better to read "to include teachers of various levels of experience." (Note: the change is made.)

Rachlin: Not a change, just a clarification. On page 457 number 2e, I would just like a definition for information network.

Becker: This came from the discussion yesterday. We thought it was a good idea that there was one point in each country where information is sent out to the participants in the seminar. So in the case of Japan it would be Professor Miwa and Mr. Sawada who would receive from us whatever information we are sending for dissemination to the Japanese delegates - or dissemination in whatever other ways they might feel are important. Similarly on the American side.

Rachlin: I guess there is a piece I don't understand in that it seems it would be just as easy for me when I'm having a secretary send it out to have sent it out to the mailing list that we have, why would I send it to one person and then funnel from there?

Becker: As I recall, though only vaguely, in the proposal yesterday part of the rationale had to do with how much money is involved in mailing costs and that sort of thing.

Wilson: In my view this should not preclude Sid sending everyone on his mailing list his vita or whatever he has ready for us. It doesn't preclude that level of communications. It shouldn't. But on the other hand, the need is to facilitate communication and if having a focal point in each country would facilitate that, then that's what we should be pursuing. Let me ask also if, we have sort of dropped what we've said above about the Japan-U.S. Joint Study Group for Mathematical Problem Solving. Is that part of the informal network idea? Or is that something separate?

Clarkson: It appears to me that the paragraph we just discussed should be moved beneath the heading that says "Proposals for Further Action."

Silver: Can we explain the nature of the Joint Study Group as you see it, what this item means and is this the U.S.-Japan Joint Study Group or will it be some subset of the participants of this meeting? Is this a matter worth discussing at this point?

Becker: I have some thoughts about it but I would like to hear from others also. The concept is that it provides a somewhat formal manner for interaction between researchers in both countries. What I see happening is that we will come up with some suggestions for joint research that may be done and how that research might be organized, the results communicated, and so on through the structure of the Joint Study Group. The way I see the Study Group, in the beginning, is this group of people.

A year from now it may be more people or it might be fewer than the number of people in this room. That remains to be seen. What unfolds after this meeting will say a lot about the nature of the group. One role of the focal points in each country is that, as we get a little further along in doing research, one of the things that has to occur is that both sides will exchange, for example, suggestions of specific problems that will be used in research, the Japanese researchers and Japanese children, the American researchers and American children, and we have to provide to each side concrete specific evidence of ideas and materials that will be implemented in the research.

Wilson: A couple of points. My first question, does it include this group or a larger group? My feeling is that if it doesn't include a larger group then it won't accomplish the larger aims. The second would be that the study group should have as its focus enhancing of communications rather than the management of studies. In the management of studies our job is to help. The study group should help facilitate or encourage but not in any sense get interested in the position of brokering. But, in fact, if Hashimoto and I want to collaborate on a study we should get on with it and let the other people know about it, but let's not have the Study Group in a position for us to ask permission to do that or work through its mechanism. That's my view.

Becker: In a sense I would see the group continuing what is already started. For example, people in the American delegation have received a lot of information from the Japanese side, copies of papers of Japanese scholars and that sort of thing. But I quite agree with Professor Wilson that, with respect to any people who want to communicate and initiate studies, the function of the group should be to encourage that.

Whitman: I wonder whether we were restricting ourselves only to mathematical problem solving as the name implies or whether now we are encompassing all of mathematics education?

Becker: Professor Whitman, do you have a suggestion?

Whitman: No, I don't. I was just wondering about it, what we should do. I'm not quite sure. I can't anticipate the volume that we are talking about. If the volume is gigantic, I can see where you might want to restrict things somewhat, but if the volume is not, then we don't want to restrict it.

Rachlin: You have already anticipated that you will need 100 copies of the proceedings for the Japanese group and 200 for the American group. Is that an indication of how many people you anticipate being in the Study Group then after those go out?

Becker: I have no idea. I think it all remains to be seen. If there are among us colleagues from Japan and America who decide they are going to begin the process of communication and interaction in doing research, that represents good potential for the group to grow. It depends a lot on what we do after this meeting.

Rachlin: Then my reaction to the question you were asking was that I would recommend changing Japan-U.S. Joint Study Group for Mathematical Problem Solving to be for mathematics education, and having it be more of a general group with the specific part being problem solving within that group. We haven't agreed what problem solving is yet but I think we can agree we are all in math education.

Becker: So Professor Rachlin proposes that the name be changed to Japan-U.S. Joint Study Group for Mathematics Education. Is it acceptable to everyone?

Silver: What I am going to suggest now will sound like I'm disagreeing with Professor Wilson, but I think I'm not really. I understood his point to be that he did not want the formation of this group to inhibit communication and cooperation between individuals in both countries and I agree with that. Nevertheless, it seems to me that one strategy we might use at this point for going forward might be to propose the formation of this U.S.-Japan Joint Study Group for Mathematics Education with several particular focal points and studies that would be conducted by this group as a starting point and to seek funding for the formation of that group in one or the other of the countries or in both countries simultaneously. Particular areas should be identified that would be the primary foci for the initial year or two years or three years of work. And then, as time evolves, the nature of the group and the nature of the work may evolve. But it is important to try to get interim funding for a two- or three-year period for this group to begin to get these cross-cultural studies going.

To clarify, I think that the activities of the group and the funding might be sought for both the general enhancement of communication between the two countries and between mathematics education groups in both countries. Also, specific research studies need to be conducted in areas that are identified as being of interest. Let me emphasize again that the idea is not to control all research activity that might happen cooperatively between the two countries, but simply to have some specific foci that the group would identify as being the ones we wish to attack first.

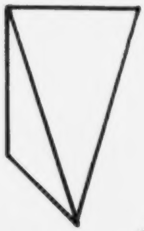
Becker: Do we have some specific suggestions for areas of research that we should pursue on both sides?

Nohda: What is important now is to identify which problems we would like to approach and to clearly delineate and define what these problem areas are and to get that started. I'm talking about not the problems for the math educators but the problems that students have, and so there are implications naturally for the curriculum. So when I talk about problems, it's not the problems that we have as math educators, I'm thinking about the kinds of problems that we may want to give the students. What I think is important is to look at the difficulties the children have when they solve a problem. What we would like to see is some Americans and some Japanese working on this problem of analyzing the difficulties the students have and see what ways we can lead them to better problem solving. One of the objectives we have in mind is how to help youngsters to develop this problem solving ability. We are asking about the possibility that we might at this time select certain aspects of the general problem of helping students to solve problems. For example, pattern finding is one aspect of this whole problem of teaching the youngsters to solve problems and maybe we can set up a common problem that one American investigator and one Japanese investigator might work on cooperatively, each in his own country, and communicating their results and so on. We might propose the investigators may want to pair off according to their interests, those interested in pattern finding or in other aspects. By this time, I guess you know, for example, Professor Travis and I have similar interests. So, would it be possible at this point to maybe start the investigation of this and make the results known through this body for example?

Becker: Thank you, Professor Nohda. Other reactions?

Sugiyama: This seminar has, of course, involved itself with problem solving, but a question that I have is whether the finding of solution to problems is enough? Is the bottom line getting solutions to problems? Is problem solving an end in itself? Or is problem solving something that we are using to achieve a greater goal? What I would like to do is to see some research concerning what's the purpose of problem solving. Is it a goal in itself or is there a larger goal of mathematical learning? Or is there something even beyond mathematical learning? I would like to do some kind of research that would maybe lead to that sort of problem and investigating where we are going.

Becker: I'm interested in learning more about how children solve non-routine problems. I think we have very little good information about that. I am also interested in generating good non-routine problems that can be used as part of problem solving instruction in the classroom. I'll give you an example:



How is this thing made? Ask children at the junior high school level to examine this, to see if they can visualize how the thing was made and then come to realize that it's made from a rectangualr piece of paper which has been made into a cylinder and then fastened at both ends like this. There are interesting questions about spatial visualization that can be asked. We can ask what will be the shape of the cross sections if a plane cuts that object in the middle parallel to the top and to the bottom? It's a square. What will the cross section look like if it's cut parallel but not through the middle? It's a rectangle. What will the cross section look like if the plane cuts the object at an angle? It's a trapezoid. What is the surface area? What is the volume of it?

Hashimoto: I might mention that Professors Becker and Shimada attended the previous U.S.-Japan Seminar. So when such kinds of cooperative works were proposed, what kind of difficulties ensued? According to your experience, would you like to explain a little about this?

Becker: Professor Hashimoto is asking about the last U.S.-Japan Seminar on Math Education. Following it, was there a model for communication, for collaborative research that we might follow here?

Shimada: At the end of the last seminar, several groups of scholars with a similar interest from the both sides were formed, and each group decided how to proceed from there. But given the nature of such informal groups without a good mechanism of communication, there was not much follow-up. As more time elapsed, less got done and less communication took place. Of the groups formed at that time, not a single one exists today, though some continued their work for a little longer than the others. Of course, I do not mean we individually lost touch with each other.

And so, if the present idea is to be continued, it seems to be important to have some liaison mechanism established and to communicate with each other in regular intervals.

Clarkson: I feel very sad at this time that we do not have another day or even another hour to do what Professor Nohda suggested; that is, to really seriously sit with others and talk about what we might be interested in doing. I believe that part of the problem with coming to a meeting like this is that at the exact time that we've gotten to know each other and each other's interests, we have to leave. If there were enough time for some sort of problem formulation, even if it were informal, we would more likely be able to do a better job.

Becker: Well, there is a little time left yet: we have five minutes here this afternoon and then after the farewell party tonight. There may be time tonight when we can talk informally with each other. Each of us knows the Japanese investigators with the same interests and some plans might be made.

Clarkson: I think it would be very important for us to take advantage of that because it's not very often that we will have a chance to all be together and talk to each other. Long distance calls and writing are not as effective.

Whitman: I wanted to ask Professor Nohda what he thought of the idea of developing a pool of problems that might be used on both sides, so that we wouldn't get overextended as far as trying to translate the problem. We could maybe kind of reduce the variables we will be looking at.

Nohda: I am interested in these problems and am now examining the problems from Europe and ancient China - looking at both ancient and modern problems. So I am interested in getting these problems together so that we have a source of problems to use in our research.

Shimada: What Dr. Nohda mentioned just now seems to be one of the suitable themes which could be pursued through the proposed network. For us, perhaps not just for me only, it becomes a problem in itself to write a scientific communication in English other than letters on daily affairs or business. We communicate in the form of a short paper or report when we have something that is completed, but this is no casual communication such as "I have got this new idea which I will just jot down and I am trying it somewhere," etc. But for the development of research, this kind of casual information exchange seems as important as the formal ones. And I think this is a good chance for us Japanese to foster such habits of casual communications with U.S. scholars through this network.

Mr. Sawada is the only one among us who has an assistant who can do that kind of thing for him. Everybody else has to do their own typing.

Wilson: I certainly endorse the spirit of what you are saying and feel that's very important. I have a couple of specific proposals. One of them is that we look ahead to 1988 when there will be several people from Japan and several from the U.S. at the ICME 6 meetings. I will not be there myself, but I know many in this room will, and look ahead to at least an informal meeting of the U.S.-Japan Joint Study Group or others who may be interested but perhaps identifying Becker and Miwa to make sure we are on the program for that or someone else from the group. That's one proposal. Second, we need to be alert to others who already have work going on with Japan and the U.S., such as Dr. Whitman with the report that she gave us. We didn't discuss it anywhere in the session but it's certainly an indication of ongoing interest that is already there. We've talked about Jack Easley's work but Jack wasn't here. Perhaps we need to know more about that. That's an example of existing work. The third one, to which I made reference in one of the discussion sessions that Professor Nohda's and Professor Miwa's student will be with me this next year and I would like to have a set of Japanese textbooks. How do I get them? I would send some elementary textbooks to someone in Japan, but I can't send sets to everyone. The question is to whom do I send them or where? I feel somehow, not being specific, that Mr. Sekaguchi and our people at Georgia will be able to have an exchange about textbook materials and compare and contrast them. We certainly have the group, this would be something informal to try.

Sawada: I hate to bring this matter up, I refer to #1ℓ, that Japan-U.S. Joint Study Group for Mathematical Problem Solving which has been changed without much objection to Japan-U.S. Joint Study Group For Mathematics Education, but please keep in mind that this particular seminar, this particular group has gathered to work on mathematical problem solving and that we are trying to formulate a paper based on this conference and this again relates to #2e, the matter of networking. The question that comes to mind is until what time is this network that has to do with math problem solving going to continue? Is it an ongoing thing, does it end after this particular conference? The reason I ask is because I have my own Joint U.S.-Japan Education Group going and that I am sure that others have various connections, groups, or similar organizations organized independently. We are not the only one, so I would like to narrow this particular discussion, this seminar, this group back down to mathematical problem solving.

Becker: Do you have reactions to Mr. Sawada's suggestion?

Rachlin: Not so much a reaction as some clarification. I wasn't aware of another study group and would like to know more about it.

Sawada: The particular group I made mention of is a much larger group organized or funded at the governmental levels of both the U.S. and Japan and it concerns itself with matters of mathematics, science and education in general. One goal of this much larger group is to promote cooperation among educators in both countries. Mathematics is one area in which much work has been done and Professor Miwa and I happen to serve on the group.

Becker: I dislike interjecting here, but I think we have to stop the proceedings for the afternoon. I would like you to remain where you are while I call Mr. McMahon and his staff in just to express our appreciation. So if you can wait for just a moment...

Here is Mr. McMahon and some of his staff. We all know Mr. McMahon who is Chief of the Logistics Section of the East-West Center. This is Rowena to his left who has been working very closely with us and Margo and Lee. There are also several other workers in the office. Norma has provided all the food and looked after those needs for us. Marshall Kingsbury has been working on the technical side of things up in the back all this time. Tammy Lewis has been very helpful also. She is not here right now.

Jim McMahon has been very helpful from the very start. From the time we knew that we would be organizing the meeting here at the East-West Center, he is the person with whom I have had contact on the phone and through the mail. His staff has really put everything together. Rowena has been absolutely excellent in all respects, accepting all of our requests right down to the last minute of looking into the postcard problem. All the others have been very cooperative and helpful and we want to take just a moment to express our appreciation to all of you for your help. This is interesting for Margo since Margo is the one who went to the florist to pick up the flowers. She went to class at 11:30 a.m. and left class to pick up the flowers. On behalf of both the Japanese and the American delegations, we thank you very much and want to present these flowers as a token of our deep appreciation.

McMahon: I want to thank you all on behalf not only of the three lovely young ladies who are here with me, but also on behalf of the rest of our group. I can't think of a better group that we've ever worked with, largely through Professor Becker and Professor Miwa. I want to thank the interpreters also, Mr. Kenney and Mr. Inouye, for the help that they've given to us and to the group here. Thank you all.

Becker: We also want to recognize the excellent contribution, the indispensible contribution to the seminar of the two translators. When Mr. Inouye was first contacted, in his typical humble way, he said he wasn't sure that he was proficient enough in the languages to help us out; yet we were assured by all the people who knew him that he was. And we are very happy that you joined us, Mr. Inouye. We appreciate your efforts very much. When I first contacted Professor Kenney, whose name came to us from Sid, he said he would be very happy to help us out and we've seen he's been very helpful. The two of them have worked together very nicely so we have a token of appreciation for them. Dr. King has stopped by - we appreciate that, and let me take this opportunity to thank him again for the excellent contributions of his staff and for his kind hospitality on Wednesday afternoon. Thank you very much. Dr. King has been very helpful regarding encouragement and good suggestions along the way and always working very quietly in the background. We appreciate your help very much, Dr. King. And, of course, our appreciation to Professor Miwa and all of his colleagues in the Japanese delegation. Professor Miwa has been a very effective, hard worker from "across the pond" as some say, always very quick to respond to the letters and even to a couple of phone calls. He has obviously organized his delegation in a most excellent manner. Professor Miwa, we appreciate all the help that you've given and the diligent work that you've put into this seminar.

Miwa: Our U.S.-Japan Seminar on Mathematics Education is closing now. I, as one Co-Organizer of the Seminar, would like to express my most sincere and whole-hearted thanks to all participants in the seminar and, in particular, to our two interpreters, Dr. Kenney and Mr. Inouye, and to the East-West Center staff.

At the beginning of the seminar, I expected our seminar to be successful and have excellent results. Now I am convinced that my expectation is fully satisfied. That is, I can say that our seminar has been successful, and I think all participants will agree with me.

As Professor Becker pointed out, we reaffirm the importance of problem solving in school mathematics both in the U.S. and

Japan. Now we have set the stage for further progress based upon the achievements of this seminar and the new information network for mathematical problem solving which will contribute much to our progress. I hope it will be successful.

Finally, as one of the Co-Organizers, I am responsible that the schedule of our seminar was so severe that all of us had scarcely any time to enjoy Waikiki. Please forgive me.

Thank you, and "arigato gozaimashita."

Becker· As Professor Miwa and I were walking over to lunch today, he mentioned that the seminar has been a success and he thought we should say that to all the members here. But he asked for my opinion whether, in my perception, the participants thought it was a successful seminar. I assured him that we were all unanimous in that feeling. Finally, I should mention that we think it is very significant, we say this to each of the members of the Japanese delegation, that we are aware of the fact that the Japanese have come to our country, to participate in a seminar, and all have done very well, participating in our language. We appreciate that very much. Unless Professor Miwa or anyone else has anything more to say, we will adjourn.

Silver: Before we close, I think I can speak for all the participants in this meeting in expressing our thanks to Professor Becker and Professor Miwa for their excellent work in organizing the seminar.

Becker: Thank you. We are adjourned.

U.S.-Japan Seminar on Mathematical Problem Solving

Findings and Proposals

(Final Draft)

1. Findings of the U.S.-Japan Seminar:

   a. The importance of problem solving is recognized unanimously by all participants. There may be some delicate differences among participants regarding the following:
      - What is problem solving?
      - What is a strategy?
      - Is problem solving activity a goal in itself or an important means to achieve a higher goal?

   b. Language seriously affects children's processes of thinking. Care must be taken in this and how to handle it is still open.

   c. The importance of teacher training is recognized unanimously as being important, especially having actual experience at problem solving suitable to each teacher's level. Reflecting on the process of arriving at a solution is indispensable in order to be a sympathetic helper to children who may feel difficulties in problem solving. Special attention is needed for pre-service teachers.

   d. Selection of problems should depend on the objective(s) of teaching at the time. Various kinds of problems may be useful to challenge the curiosity of students and their inquiring minds if they are well-organized. How to relate problem solving more realistically to students and to make it interesting is an important question on which we want to continue to focus.

   e. In teaching problem solving, we need to use methods to encourage inquiry and to help students to be conscious of the processes which are used for arriving at solutions. In this manner, we may teach a kind of strategy.

   f. Each side wishes to continue to learn more about how thinking and problem solving are approached in the mathematics curriculum and classrooms in the other's country.

   g. Both sides agree that evaluation of problem solving is important and should be a focus of our interest. For example, how is pattern finding evaluated?

   h. Both sides agree that a need exists to see how classroom teaching is carried out in each country.

   i. The goals of mathematics education need to be analyzed to see how problem solving fits into the framework.

   j. There is a need to further consider the question of teachers' activities in the classroom - children centered or teacher centered?

   k. Both sides agree that interaction between members of the Japanese and United States mathematics education communities is important and that collaborative research holds important potential for expanding our knowledge of learning and teaching problem solving.

1. Both sides agree that a Japan-U.S. Joint Study Group for Mathematical Problem Solving should be formed. The Joint Study Group will provide a facilitating and enhancing vehicle for formalizing and carrying out cross-cultural activities of all kinds in mathematics education. The Joint Study Group will serve to continue the friendly and scholarly collegiality generated in this seminar.

2. Proposals for further action:

   a. A series of problems should be developed, of a non-routine nature, that can be used in the classroom, in teacher training, and in research at the various levels of school and teacher training.

   b. A program of collaborative research into children's problem solving behavior needs to be planned. We should target the following populations·
      - lower elementary grade
      - middle elementary grade
      - upper elementary grade
      - one grade of the junior high school
      - one grade in the senior high school
      - students in preservice teacher education programs

   c. An exchange of inservice mathematics teachers and mathematics education researchers should be planned.

   d. An exchange of videotapes of classroom scenes on problem solving should be planned, to include teachers at various levels of experience.

   e. An information network should be established through which ideas, information, materials, and progress in cooperative, joint work can be communicated. For this purpose, Professor Miwa, Mr. Sawada (Japan) and Professors Becker and Wilson (USA) may serve as focal points with the responsibility of facilitating information/materials flow.

   f. Both countries have exceptionally good teachers. The question of what makes an exceptional teacher should be addressed within and between countries.

   g. In order to acquire a better understanding of material(s) and teaching in the mathematics curriculum of each country, the question of translation of materials should be addressed in a realistic manner:
      - consider translation of only the problem solving material in textbooks and teacher training materials from each side.
      - consider the possibility of translating test materials from each side.

   h. There is a need to pursue funding possibilities to support and carry to reality the outcomes and future plans of this seminar.

   i. We need to take care to:
      - widely disseminate the proceedings of this conference
      - look for opportunities to exchange mathematics educators from the two countries
      - explore, as possible and in a timely manner, other societal/cultural characteristics that impact on problem solving behavior; e.g., expectation of teachers and parents.

Discussion concerning Seminar Proceedings:

Papers (due to J. Becker by October 1, 1986)

Discussions - transcribed and finalized after each participant has had an opportunity to review.

Highlights of the Seminar - a short paper (3-5 pages) agreed to by participants.

Tentative publication date: July, 1987

Outline of parts of proceedings:

- Introduction
- Papers
- Discussions
- Participants/photograph
- Overview of Seminar
- Seminar findings
- Seminar follow-up activities

APPENDIX*

Examples of Problems in Mathematics for the Entrance Examinations to National Universities (Secondary Examinations) and Private Universities in Japan

I. Full Set of Problems, University of Tokyo, 1982.

Six problems for the Science section (bound for faculties of science, engineering, medicine, agriculture and pharmacy). Given time is 150 minutes. Four problems for literature section (bound for faculties of law, economics, humanities and education). Given time is 100 minutes.

Science Section

[S1] Let f be a linear transformation of xy-plane defined by a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and suppose that there exists a point P different from the origin which is mapped to P itself by f. Then prove that there exists a straight line l which does not go through the origin and which is mapped into (or onto) l by f.

[S2] Let T be a regular tetrahedron and S be a sphere of radius 1. Suppose that each edge of T is tangential to S. Then find the length of an edge of T, and furthermore find the volume of the solid portion which is exterior to T and, at the same time, interior to S.

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* This is an excerpt from a paper by H. Fujita entitled "The Present State and a Proposed Reform of Mathematical Education at Senior Secondary Level in Japan" (*Journal of Science Education in Japan*, Vol. 9, No. 2, 1985, Japan Society of Science Education. The original was presented at AG4 of ICME 5, Adelaide, 1984.), and was distributed by Professor Shimada at the Seminar for information as to the level and nature of mathematics problems in university entrance examinations in Japan.

[S3] There be given two moving points A and B in xy-plane; A moves along the subarc lying in the first quadrant of the unit circle with radius 1 and center at the origin O, while B moves on the x-axis in such a way that the length of the segment AB is kept equal to 1. Moreover, the point C of intersection of the segment and the circle is different from A or B. Under these circumstances, answer the following questions·

(1) Determine the range of $\theta = \angle AOB$.

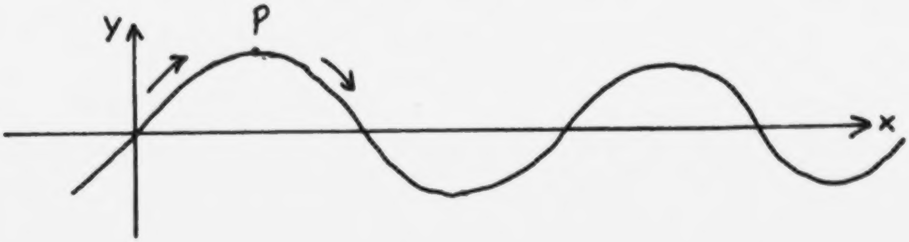
(2) Express the length of BC in terms of θ.

(3) Denoting the middle point of OB by M, determine the range of the length of CM.

[S4] There moves a point P along the curve $y = \sin x$ in xy-plane from left to right as indicated in the figure. Assume that the speed of P is of a constant value V $(V > 0)$. Then find the maximum of the magnitude $|\vec{\alpha}|$ of the acceleration vector $\vec{\alpha}$ of P. Here the speed of P means the magnitude of the velocity vector

$\vec{v} = (v_1, v_2)$ of P and $\vec{\alpha}$ is given by

$\vec{\alpha} = \left(\frac{dv_1}{dt}, \frac{dv_2}{dt}\right)$, t being the time variable.



[S5] Find the volume of the solid in xyz-space which is the set of all points with coordinates x, y, z subject to

$0 \leq z \leq 1 + x - y - 3(x - y)y, \quad 0 \leq y \leq 1$ and

$y \leq x \leq y + 1$.

[S6] Firstly a dice is placed with its 1-spot face at the top. Then we change the position of the dice by repeated rotations. At each rotation, the angle of rotation is 90 degrees and the axis of rotation is a straight line connecting centers of a randomly chosen pair of square faces parallel to each other. Moreover, assume that any choice of possible axes is equally likely, and for each chosen axis any one of two possible directions of rotation is equally likely.

Let p_n, q_n and r_n denote the probabilities that the face of 1-spot is located on the top, on some of lateral sides and on the bottom just after the n-th rotation, respectively.

(1) Find p_1, q_1 r_1.

(2) Express p_n, q_n, r_n in terms of p_{n-1}, q_{n-1}, r_{n-1}.

(3) Find $p = \lim_{n\to\infty} p_n$, $q = \lim_{n\to\infty} q_n$, $r = \lim_{n\to\infty} r_n$.

Literature Section

[L1] Two points A, B are fixed in a plane, where the length $\overline{AB}$ of the segment AB is equal to $2(\sqrt{3}+1)$. Now consider three points P, Q, R which move in this plane keeping the relations $\overline{AP} = \overline{PQ} = 2$ and $\overline{QR} = \overline{RB} = \sqrt{2}$. Find the domain S in the plane which is a set of all points Q can reach, and also find the area of S.

[L2] We denote the following three variable points on the curve $y = x^2$ in xy-plane by A, B, C in accordance to the order of their x-coordinates (the x-coordinate of A is the smallest). While they move, the difference of the x-coordinates of A and B is kept equal to a (a is a positive constant) and the difference of the x-coordinates of B and C is kept equal to 1. Now, express the x-coordinate of A by a for the case where $\angle CAB$ attains its maximum. Moreover, find the value of a such that $\angle ABC$ becomes a rectangle when $\angle CAB$ attains its maximum.

[L3] Consider 4 roots of the equation $x^4 + ax^2 + b = 0$ where a and b are integers. We are given approximate values of these 4 roots as -3.45, -0.61, 0.54, 3.42, where the absolute value of the error of any of these approximate values does not exceed 0.05. Then write down numerical values of the exact roots up to the second decimal places.

[L4] Let

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Starting with $P_0 = (1, 1)$, we generate sequences of points P_0, P_1, P_2, . . . in xy-plane by the following procedures, where (x_n, y_n) stands for the coordinates of P_n.

a) If $x_n + y_n \geq \frac{1}{100}$, then (x_{n+1}, y_{n+1}) is given

either by $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ or by $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = B \begin{pmatrix} x_n \\ y_n \end{pmatrix}$.

b) If $x_n + y_n < \frac{1}{100}$, then (x_{n+1}, y_{n+1}) is given by

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

In this way a number of sequences of points are generated. Now answer the following questions.

(1) Find all points possible as P_2, and show them graphically.

(2) Express $x_n + y_n$ through n.

(3) How many points are possible as P_n?

II. Selected Problems.

[1] (Kyoto University, 1979. Science Section)

Find polynomials $P_1(x)$, $P_2(x)$, $P_3(x)$ of which the coefficient of the highest degree term is equal to 1 and which satisfy the following conditions, respectively:

(1) $P_1(x)$ is linear and $\int_{-1}^{1} P_1(x)C\,dx = 0$ for any constant C.

(2) $P_2(x)$ is quadratic, and $\int_{-1}^{1} P_2(x)f(x)dx = 0$ for any polynomials $f(x)$ with degree equal to or less than 1.

(3) $P_3(x)$ is cubic, and $\int_{-1}^{1} P_3(x)f(x)dx = 0$ for any polynomial $f(x)$ with degree equal to or less than 2.

[2] (Kyoto University, 1979, Science Section)

P is a moving point in xy-plane. At time t, the coordinates (x, y) of P is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{-at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where a, b, c_1, c_2 are constants with $a > 0$, $b > 0$. The point $C(c_1, c_2)$ is assumed to be different from the origin O(0, 0).

(1) Show that the angle between the velocity vector of P and the radial vector OP is constant.

(2) After starting from $C_0 = C$, the point P intersects the segement OC for the first time at C_1, for the second time at C_2, . . ., for the k-th time at C_k. Now show that the length of the arc between C_k and C_{k+1} of the orbit of P forms a geometric sequence, namely that the length of $\overset{\frown}{C_0C_1}$, $\overset{\frown}{C_1C_2}$, . . ., $\overset{\frown}{C_kC_{k+1}}$, . . . is a geometric sequence.

[3] (Kyoto University, 1980, Literature Section)

Suppose that $f(x)$ is a polynomial of degree n and n $n \geq 2$. Then show that

$$f(x) + f(x+1) - 2\int_0^1 f(x+t)dt \quad \text{is of degree } n - 2.$$

[4] (Kyoto University, 1980, Literature Section).

Let S be a set $S = \{a_1, a_2, \ldots, a_n\}$ of n distinct positive numbers with the following property: if we take any two different elements a_i, a_j from S, then either $a_i - a_j$ or $a_j - a_i$ belongs to S.

Now show that by a suitable rearrangement of the order, a_1, $a_2, \ldots, a_n$ can form an arithmetic progression.

[5] (Waseda University, 1982, Faculty of Science and Engineering). Let R be the image of the following figure in (1)-(3) by the linear transformation $\begin{matrix} x' \\ y' \end{matrix} = \begin{matrix} 1 & 3 \\ 2 & k \end{matrix} \begin{matrix} x \\ y \end{matrix}$, respectively.

Determine which will be R, a point, a straight line or the whole plane. If R is a point or a straight line, then write its coordinate or equation, respectively.

(1) straight line: $x + 2y = 0$
(2) straight line: $x + 2y - 3 = 0$
(3) whole plane.

[6] (Waseda University, 1982, Faculty of Science and Engineering). Suppose that the point $P(x, y)$ in xy-plane moves on the unit circle with center at the origin. Then find all points where the maximum of $z = x^3 + y^3$ is attained. Also, write the maximum value of z.

[7] (Waseda University, 1982, Faculty of Science and Engineering). The coordinates x, y, z of a variable point P in the space are regarded as functions $x(t)$, $y(t)$, $z(t)$ of time t $(0 \leq t \leq \frac{1}{2})$

and are subject to the conditions:

(i) $\dfrac{dx}{dt} = z - y$, $\dfrac{dy}{dt} = x - z$, $\dfrac{dz}{dt} = y - x$

(ii) $(x(0), y(0), z(0)) = (1, -1, 0)$,

(iii) $x(t) \quad y(t), \ z(t) \leq 0$.

Then answer the following questions.

(1) Find $f(t) = x(t) + y(t) + z(t)$.
(2) Find $g(t) = (x(t))^2 + (y(t))^2 + (z(t))^2$.
(3) By $\theta(t)$ we denote the angle between the vector (1, -1, 0) and the vector $(x(t), y(t), z(t))$. Express $\sin\theta(t)$ in terms of $z(t)$.
(4) Find $\theta(t)$.

[8] (Waseda University, 1982, Faculty of Education).

Let α and β be two planes in xyz-space such that α contains $(0, 0, 0)$, $(1, 1, 1)$, $(0, 1, -1)$ and contains $(0, 0, 0)$, $(-1, 1, 1)$, $(1, 0, 1)$. Find the point of intersection of the plane β and the straight line which passes through $P(a, b, c)$ and perpendicular to the plane α .

[9] (Waseda University, 1982, Faculty of Education).

Let $f(x)$ and $g(x)$ be differentiable functions defined on $0 < x < \pi$ and suppose that f and g satisfy the conditions

$$f'g = \sin x + \cos x, \quad fg' = -\sin x, \quad f(\tfrac{\pi}{2})g(\tfrac{\pi}{2}) = 1, \quad \lim_{x \to 0} g(x) = -1.$$

Determine $f(x)$ and $g(x)$.